NASA TECHNICAL TRANSLATION

NASA TT F-539

NASA TT F-539

MOTION OF ARTIFICAL SATELLITES IN THE EARTH'S GRAVITATIONAL FIELD

by Yu. G. Yevtushenko, I. A. Krylov, R. F. Merzhanova, and G. V. Samoylovich

Computer Center, Academy of Sciences USSR
Moscow, 1967

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION • WASHINGTON, D. C. • JUNE 1969

NASA TT F-539

MOTION OF ARTIFICIAL SATELLITES IN THE EARTH'S GRAVITATIONAL FIELD

By Yu. G. Yevtushenko, I. A. Krylov, R. F. Merzhanova, and G. V. Samoylovich

Translation of "Dvizheniye Iskusstvennykh Sputnikov v Gravitatsionnom Pole Zemli."
Computer Center, Academy of Sciences USSR, Moscow, 1967

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

For sale by the Clearinghouse for Federal Scientific and Technical Information
Springfield, Virginia 22151 – CFSTI price $3.00

TABLE OF CONTENTS

FOREWARD

The development of methods for effectively computing the trajectories of artificial satellites in the actual field of the Earth is one of today's chief technical problems. There have probably been dozens of studies devoted to this subject. In this book, some of these methods are analyzed, described and compared. We have also included research done along these lines by the workers at the Computing Center of the Academy of Sciences of the USSR. This monograph is the collective work of Yu. G. Yevtushenko (§12 and Appendices V111 and X), I. A. Krylov (§15), R. F. Merzhanova (§13, §14) and G. V. Samoylovich, who wrote the remaining sections of the book.

Appendix V11 was written by G. V. Samoylovich and Yu. G. Yevtushenko, and all of the authors of the book contributed to compilation of §16 (comparative analysis). Most of the calculations were handled by R. F. Merhanova. Yu. G. Yevtushenko and I. A. Krylov also took part in programming and carrying out computations. The book also takes up a number of auxiliary problems relating to the nature of the Earth's potential with a systematic exposition of the given subject.

The work may be used as a reference and a textbook.

N. N. Moiseyev

FROM THE AUTHORS

/5*

The actual motion of artificial celestial bodies differs from that given by Kepler's laws due to the effect of various disturbing factors. This book is devoted to quantitative and qualitative analysis of motion which is affected by the most appreciable (at moderate distances from the Earth) of these factors--the eccentricity of the Earth's gravitational field. These problems are taken up primarily in the first two chapters. The third chapter contains an outline of several algorithms which employ analytical relations to describe the motion of artificial satellites in the gravitational field of the aspherical Earth. This chapter puts several methods at the reader's disposal, from which he may select the one which best suits the practical problem to be solved. The fourth and final chapter contains several examples illustrating application of the material given in preceding sections.

The authors started out with the goal of writing a book which would be of use to practioners: engineers and scientific collaborators involved to some extent in studying and computing the motion of artificial Earth satellites. This goal influenced the selection of material, manner of exposition and structure of the book. More specifically, an attempt was made to present the basic material, comprising nineteen chapters, as simply as possible (but with sufficient rigor) without losing sight of the chief purpose--practical application.

The appendices to the chapters contain derivations of the relationships and proofs of some of the facts cited in the main text, some auxiliary theoretical data, and also the values of a number of constants needed in practical calculations.

Thus, the basic material is sufficient for a first acquaintance with the problems treated in this book. The appendices should be used for a deeper study of the pertinent sections.

In completing this work, the authors received assistance and advice from Professor N. N. Moiseyev, who instigated both the writing of this book as well as the entire series "Mathematical Methods in the Dynamics /6
of Space Vehicles". We were assisted by Ye. P. Aksenov and A. A. Orlov in starting and programming the algorithms given in §15 and 14 on a digital computer. A great deal of constructive criticism during reading of the manuscript was rendered by O. A. Chembrovskiy and L. P. Pellinen, resulting specifically in improvement of the first chapter. V. N. Lavrik handled the set of computations given in the second chapter as well as programming of the corresponding problems on a digital computer. A. F.

---

*Numbers in the margin indicate pagination in the foreign text.

Shutkina meticulously handled a great quantity of graphic material. The authors give their sincere thanks to all these comrades.

"Having experienced the torment of /7
thirst, I endeavoured to dig a well
that others might draw from."

E. Seton-Thompson

CUT ALONG THIS LINE

FOLD LINE

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION
WASHINGTON, D.C. 20546

OFFICIAL BUSINESS

POSTAGE AND FEES PAID
NATIONAL AERONAUTICS & SPACE ADMINISTRATION

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION
CODE USS-T
WASHINGTON, D.C. 20546

NASA TTF No.
539

FOLD LINE

CUT ALONG THIS LINE

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

TECHNICAL TRANSLATION EVALUATION

Budget Bureau No. 104-R037
Approval Expires: Sept. 30, 1969

TO:
THE USERS OF THIS TRANSLATION →

NASA TTF NO. 539

MAINTAINING THE QUALITY OF NASA TRANSLATIONS REQUIRES A CONTINUOUS EVALUATION PROGRAM. PLEASE COMPLETE AND MAIL THIS FORM TO AID IN THE EVALUATION OF THE USEFULNESS AND QUALITY OF THE TRANSLATING SERVICE.

THIS PUBLICATION *(Check one or more)*

- [ ] FURNISHED VALUABLE NEW DATA OR A NEW APPROACH TO RESEARCH.
- [ ] VERIFIED INFORMATION AVAILABLE FROM OTHER SOURCES.
- [ ] FURNISHED INTERESTING BACKGROUND INFORMATION.
- [ ] OTHER *(Explain)*: ______________________

FOLD LINE — — — — — — — — — — — — — — — — **FOLD LINE**

TRANSLATION TEXT *(Check one)*

- [ ] IS TECHNICALLY ACCURATE
- [ ] IS SUFFICIENTLY ACCURATE FOR OUR PURPOSE.
- [ ] IS SATISFACTORY, BUT CONTAINS MINOR ERRORS.
- [ ] IS UNSATISFACTORY BECAUSE OF *(Check one or more)*:
  - [ ] POOR TERMINOLOGY.
  - [ ] NUMERICAL INACCURACIES.
  - [ ] INCOMPLETE TRANSLATION.
  - [ ] ILLEGIBLE SYMBOLS, TABULATIONS, OR CURVES.
  - [ ] OTHER *(Explain)*: ______________________

FOLD LINE — — — — — — — — — — — — — — — — **FOLD LINE**

REMARKS

FROM

DATE

NOTE: REMOVE THIS SHEET FROM THE PUBLICATION, FOLD AS INDICATED, STAPLE OR TAPE, AND MAIL. NO POSTAGE NECESSARY.

NASA Form 1332 AUG 66

GPO 915-777

CUT ALONG THIS LINE

CUT ALONG THIS LINE

Introduction

The equation for an artificial Earth satellite in a central gravitational field (the field of a material point or a uniform sphere) takes the form

$$\frac{d^2\vec{r}}{dt^2} = -\frac{k^2 M\vec{r}}{|\vec{r}\,|^3} = \vec{F}_c, \qquad (0.1)$$ /9

where $\vec{r}$ is the radius vector for the center of gravity of the satellite; t is the time; k^2 is the constant of attraction or the gravitational constant; M is the mass of the attracting body or the central mass.

The quantity k^2 is equal to the proportionality factor f in expression (2.1) given below. In the CGS system, $f = 6.67\cdot10^{-8}$ cm$^3\cdot$g$^{-1}\cdot$sec^{-2}.

The physical meaning of the constant $k^2 = f$ is evident from formula (2.1). It is the acceleration toward the central mass which is acquired by a body of unit mass separated by a unit of distance[1].

The motion of artificial Earth satellites calculated by law (0.1) is considered by S.S. Tokmalayeva[2]. The actual motion of satellites and space vehicles never takes place under the effect of a central force. This /10 motion is described by an equation which may be written in the form

$$d^2\vec{r}/dt^2 = \vec{F}_c + \vec{R} \qquad (0.2)$$

and is called the equation of disturbed motion.

The name is purely arbitrary and assumes that the basic force which determines the motion is central, that all remaining forces are small in comparison with the basic force, and that the motions described by equations (0.1) and (0.2) are close to one another.

This last condition is useful not only for writing equations of motion

[1] If we take as units of mass, time, and distance respectively the mass of the sun, the mean solar day, and the semi-major axis of the Earth's orbit (the so-called astronomical unit of distance), then k^2 is called the Gauss constant. According to Gauss's computations, k = 0.01720209895. Modern data give k = 0.01720209842. The value computed by Gauss is taken as the constant for greater convenience, but a correction is introduced in the value of the semi-major axis of the Earth's orbit, considering it equal to a = 1.00000003, or log a = 0.0000000013. The quantity k^2 may be equal to unity if the unit of time is taken as equal to k^{-1} = 58.132441 mean solar days.

[2] S. S. Tokmalayeva's paper will be published in this same series.

in a form which is convenient for study (equations in oscillating elements, see [1-3], but also for finding analytical (see for instance §11 of this book) or numerical solutions (for instance in the Encke method [1]).

The function R is a perturbation or disturbing force (or function) [1, 2]. The form of the perturbation functions, which may be either of "natural" origin or associated with intentional human actions (a force which controls motion), depends on the physical scheme of the problem.

In this book, which is the second in the series "Mathematical Methods in the Dynamics of Space Vehicles," we shall take up uncontrolled (ballistic) motion of a space vehicle or artificial satellite in the central gravitational field of the Earth. The investigation of this motion takes on added significance in connection with planning and determining the orbits of space craft (4). The pertinent equations (of form (0.2)) may be solved (without using numerical integration)only for special forms of the disturbing function and in a specially selected coordinate system[1]. Some methods for describing the motion of artificial satellites in an eccentric field of terrestrial attraction are considered in the third chapter. However, as a preliminary step, the nature of such trajectories is analyzed and brief consideration is given to the various forces which disturb the Keplerian orbits of artificial Earth satellites.

---

[1] Equations (0.1) may be integrated in closed form with the use of the first integrals (See [1-3]).

"The Earth is not at all an ellipsoid of revolution, but rather shows a wavelike deviation from the ellipsoid which describes it as a whole." /11

K. F. Gauss

Chapter One

THE FORCE FIELD AND SHAPE OF THE EARTH

§1. The Forces Which Act on an Artificial Earth Satellite

The motion of an uncontrolled space vehicle under the effect of a central attractive force alone takes place in a bounded region and conforms to Kepler's laws, assuming that the inequality

$$V^2 < 2\mu/r$$

is satisfied at each point of the trajectory, where V is the absolute velocity of the vehicle, r is the distance from the vehicle to the center of the attracting mass, and μ is the product of the gravitational constant and the mass of the attracting body (see §2).

In reality, the trajectory of such a vehicle is affected, not only by the central gravitational force, but also by disturbing effects which cause the motion to deviate from Keplerian motion. The disturbing forces differ in value, and the principal forces may be isolated for further analysis. These disturbing factors are the eccentricity of the gravitational field, the effect of the ambient medium, the effect of pressure from solar radiation, the effect of magnetic fields, and the relativistic effect.

Let us examine the nature and magnitude of these effects individually.

Effect of the Eccentricity of the Gravitational Field /12

The eccentricity of the gravitational field in which an artificial Earth satellite or any other space vehicle moves is due to the effect of the gravitational fields of other heavenly bodies and the eccentricity of the Earth's field itself. The effect of eccentricity in the Earth's gravitational field will be analyzed in more detail later on. For now we shall examine disturbances due to the effect of the moon and sun, since theirs are the only effects out of the entire set of heavenly bodies which can be satisfactorily examined.

The accelerations which correspond to the effect of these forces (in an inertial geocentric coordinate system) are defined by the following equations:

$$\left.\begin{aligned}
\ddot{x} &= f m_S\left(\frac{x_{KS}}{r_{KS}^3} - \frac{x_{TS}}{r_{TS}^3}\right) + f m_L\left(\frac{x_{KL}}{r_{KL}^3} - \frac{x_{TL}}{r_{TL}^3}\right);\\
\ddot{y} &= f m_S\left(\frac{y_{KS}}{r_{KS}^3} - \frac{y_{TS}}{r_{TS}^3}\right) + f m_L\left(\frac{y_{KL}}{r_{KL}^3} - \frac{y_{TL}}{r_{TL}^3}\right);\\
\ddot{z} &= f m_S\left(\frac{z_{KS}}{r_{KS}^3} - \frac{z_{TS}}{r_{TS}^3}\right) + f m_L\left(\frac{z_{KL}}{r_{KL}^3} - \frac{z_{TL}}{r_{TL}^3}\right),
\end{aligned}\right\} \qquad (1.1)$$

where m_S is the relative mass of the sun with respect to the Earth ($m_S = 0.0332448 \cdot 10^6$); m_L is the relative mass of the moon ($m_L = 0.0122888$). The subscripts are: K = artificial Earth satellite, S = sun, L = moon. T = Earth.

These disturbances result in rotation (so-called precession) of the plane of motion of the satellite with respect to the pole of the ecliptic with angular velocities $\dot{\Omega}_S$ and $\dot{\Omega}_L$, and also change the modulus of the focal radius for the orbit of the artificial satellite (δr_S and δr_L).

The following estimates may be given for the functions δr and Ω. For a circular orbit 800 km from the Earth (for a single revolution):

$$\delta r_{S\,\max} = 25.6 \text{ cm};$$
$$\delta r_{L\,\max} = 57.5 \text{ cm}.$$

The total deflection due to the effects of sun and moon on a single revolution

$$\delta r_{L,S} < 83 \text{ cm}.$$

Under the same conditions, an artificial Earth satellite in a "24-hour" orbit (6.61 Earth radii) is subjected to the disturbances: /13

$$\delta r_{L\ max} = 783\ m;$$
$$\delta r_{S\ max} = 313\ m;$$
$$\delta r_{S,L\ max} < 1096\ m.$$

The rate of motion of the ascending node with respect to the pole of the ecliptic is equal to

$$\dot{\Omega}_L = 0.63 \cdot 10^{-10} \sec^{-1};$$
$$\dot{\Omega}_S = 0.28 \cdot 10^{-10} \sec^{-1},$$

and with respect to the pole of the Earth

$$\dot{\Omega}_{L,S} = 0.835 \cdot 10^{-10} \sec^{-1}.$$

For artificial satellites located in a "24-hour" orbit, the precession of the plane with respect to the planet's pole is equal to

$$\dot{\Omega}_S = 4.05 \cdot 10^{-10} \sec^{-1};$$
$$\dot{\Omega}_L = 10.2 \cdot 10^{-10} \sec^{-1};$$
$$\dot{\Omega}_{L,S} < 14 \cdot 10^{-10} \sec^{-1}.$$

It may be pointed out for comparison that precession of the plane of the orbit due to the first degree of polar flattening of the Earth (See § 5) is twice as great in this case

$$\dot{\Omega}_T < 30 \cdot 10^{-10} \sec^{-1}.$$

According to data [5], the precession of the orbital plane with respect to the pole of the ecliptic depends on the length of the semi-major axis of the orbit as follows:

$$\dot{\Omega}_T \sim a^{-7/2};$$
$$\dot{\Omega}_L \sim a^{3/2}.$$

Thus, $\dot{\Omega}_T = \dot{\Omega}_L$ at a distance equal to somewhat more than seven Earth radii.

The disturbing effect of the moon (and in the general case, of any celestial body) in certain cases may considerably alter the pattern of motion of the artificial Earth satellite. Interesting results along these lines were obtained by M. L. Lidov [6], the most important of which relates to the fact that the lifetime of the satellite and that of the disturbing body. At the distance mentioned above (about 7 Earth radii), this effect can no longer be disregarded in computations. For instance, according to J. Kozai's computations [7], the disturbing effect of the moon shortened the lifetime of satellite 1959 δ_2 "Explorer VI", by a factor of more than 10 (orbital parameters: a_0 = 43,446 km, e_0 = 0.7604, i_0 = 47°.10). Actual determination of the orbital evolution of satellite 1958 β_2 (parameters: altitude above the Earth's surface at apogee h_A = 3,948 km, altitude at perigee h_Π = 658 km, i_0 = 34°.3, e_0 = 0.19, Keplerian period of rotation $T_K = 134^m.18$, weight G - 2 kg), "Vanguard 1", due to the disturbing effect of the sun, showed the following values: $\dot{\Omega}_S$ = 0.18°/yr, δr_S = 56.6 cm. The effect of the moon on the orbital evolution was 2.2 times as great. /14

Thus, if we disregard the disturbing effect of the moon and the sun when considering artificial satellites moving over a period of several days at a moderate distance from the Earth (3-4 thousand km above the surface), we introduce an error of several hundred meters in the position of these satellites.

For satellites with an apogee at about 40,000 km over this same time period, the error reaches several dozen kilometers (this error is due chiefly to inaccuracy in locating the plane of the orbit; the error due to inaccuracy in determining the focal radius is only about one kilometer).

Effect of the Ambient Medium

As distinct from the conservative[1] effect of the masses of the moon

---

[1] A mechanical system in which the total energy remains constant is called conservative. If the potential energy of the system u depends only on the coordinates (e.g., x, y, z), then the forces in the conservative system are defined in the form $F_x = -\partial u/\partial x$, $F_y = -\partial u/\partial y$, $F_z = -\partial u/\partial z$ (this condition is necessary and sufficient). In contrast to the conservative system, the total mechanical energy decreases continuously in the dissipative system with conversion to other forms of energy (chiefly thermal).

and sun, the ambient medium has a dissipative effect on satellite motion. The energy of the moving body and its altitude above the Earth decrease continually and the satellite, as it enters the dense layers of the atmosphere, goes into a steep descent trajectory. The braking effect of the atmosphere is characterized by the deceleration of the body, which may be computed from the well-known formula

$$\vec{A} = -B\rho V_{REL}^{2}\vec{V}_{REL}^{\circ}. \qquad (1.2)$$

Here, the notation V_{REL} = the velocity of the satellite with respect to the atmosphere; $\vec{V}_{REL}^{\circ}$ is the unit vector of relative velocity; ρ is the mass density of the atmosphere; $B = C_x S/2m$ is the ballistic coefficient of the satel- /15 lite; C_x is the aerodynamic drag coefficient which depends on the shape of the body and the flow conditions; S is a characteristic area (e.g. the middle cross-sectional area) to which the coefficient C_x is reduced; m is the mass of the body.

The coefficient of aerodynamic drag of a satellite depends on a number of factors (the geometric shape of the object, its orientation with respect to the vector of relative velocity, atmospheric temperature, conditions of interaction between the molecules and atoms of the upper atmosphere and the surface of the object, etc.). At present, the value of C_x may be reliably determined only for bodies of the simplest geometric shape. More specifically, conical and convex bodies have a drag coefficient equal to $C_x \simeq$ $\simeq 1.7 - 2.1$ [8][1]

The ballistic coefficient is a comprehensive aerodynamic characteristic for space vehicles moving in the upper layers of the atmosphere.

Unfortunately, not only is the value of B uncertain, but so is the

---

[1] Since flow around bodies at altitudes of more than 150 - 200 km takes place in the free-molecular state, diffuse reflection is the most probable mechanism of interaction between the atmosphere and the body. The Newtonian theory based on this premise may be used for computing the value of C_x, which is equal to two and is independent of the shape of the body. More accurate methods of determining coefficients of aerodynamic drag (e.g. [8]) show some deviation from this value.

density ρ of the upper atmosphere[1]. We shall not take up this problem in detail (of the new works in this area see for instance [11]); it is sufficient to state that the variations in density which depend on the diurnal rotation of the Earth reach 100% [12]; the same magnitude of fluctuation in density (with a period of fluctuation of less than half a day) may result from a change in solar activity. Besides, the change in density depends on the annual rotation of the Earth and the geographic latitude of the locality. /16

Since the relationship between the parameters of the upper atmosphere and solar activity is not presently known with sufficient accuracy, and we cannot reliably predict the change in ρ as a function of other parameters, the density of the upper atmosphere is actually a random function of many variables. The effect of random changes in ρ, on the accuracy of determining satellite motion may be estimated, if only by comparing the results of computations for various models of the atmosphere.

Let us cite some computational results in order to give an idea of the magnitude of disturbances due to the atmosphere[2].

For a circular satellite (at a distance of 225 km), the change in longitude of the ascending node and the inclination of the orbit reach the following values in a 24-hour period:

$$\Delta\Omega \simeq 1'', \quad \Delta i \simeq 10''.$$

In this and subsequent examples, we have used S. K. Mitra's model of the atmosphere [18] corrected somewhat to bring it into conformity with the latest data.

For a satellite with initial eccentricity $e_0 = 0.0499$ ($h_A = 1{,}000$ km, $h_\Pi = 300$ km), these quantities remain practically constant over a period of

[1] Since the atmosphere rotates with the earth, the artificial satellite is affected not only by the dissipative force in the plane of the orbit, but by a disturbing force normal to the plane which causes a change in the longitude of the descending node and the inclination. For instance, as is shown in [9] and [10], the rotation of the atmosphere caused a secular variation in the orbital inclination of satellite 1957 β "Sputnik 2" by $4\cdot10^{-4}$ deg/day (by calculations) or $1\cdot10^{-3}$ deg/day (actual). We shall consider only a nonrotating atmosphere in this section.

[2] Of the works on determining the effect of atmospheric drag on satellite motion, we should mention the articles by D. Ye. Okhotsimskiy, T. M. Eneyev and G. P. Taratynova [13], G. P. Taratynova [14, 15] and P. Ye. El'yasberg [16, 17].

several days. Changes in the geometric dimensions of a circular orbit (h=250 km) due to the effect of the atmosphere become comparable after only ten revolutions with disturbance due to first-order polar flattening of the Earth (i.e. they are equal to approximately 20 km). For an elliptical orbit (h_A = 1,000 km, h_Π = 300 km), the change over this same interval is no greater than 0.5 km.

There is a more appreciable reduction in the draconic period of revolution of the satellite T_Ω[1]. For the given circular and elliptical satellites, these changes are equal to 110 and 3 sec respectively by the end of the tenth revolution. /17

Thus, the ambient medium is an appreciable factor which affects the motion of artificial Earth satellites in the range of altitudes below 300 km from the surface of the Earth. In many instances (especially when studying motion over a comparatively long time interval) this effect must be taken into account. Naturally, all problems associated with satellite lifetime are due in the final analysis to the effect of the atmosphere. However, in the general case, the necessity for accounting for the effect of the atmosphere depends on the specific formulation and initial conditions of each given problem.

Effect of Magnetic Fields. The Relativistic Effect

The electrical systems installed in satellites and the magnetic field induced by these systems make the motion of the satellite sensitive to the Earth's magnetic field and to the local random variations in this field. There has not been sufficient research on the problem of interaction between the Earth's magnetic field and those of the satellites (V. V. Beletskiy's paper [20] takes up the effect which this factor has on satellite motion with respect to the center of gravity); however, it may be assumed that the effect of this interaction will be several times less than that of the upper atmosphere at moderate distances from the Earth (700 - 3,000 km).

Since the effects associated with the theory of relativity become appreciable when an object is moving close to the speed of light, it may be assumed ahead of time that the relativistic effect has only a slight influence on the motion of an artificial Earth satellite over a comparatively short time interval. However, let us examine this effect in more detail for the sake of generality in our analysis.

The relativistic effect in determination of artificial satellite motion is associated with the fact that the trajectory of a satellite (writing out the equations of motion, giving the initial conditions) is determined by methods which are now well-known in inertial Galilean space. Actually, according to the general theory of relativity, any coordinate system which is

[1] The draconic period is defined as the interval of time between two successive transits of the satellite through an ascending node [19].

inertial from the standpoint of Newtonian mechanics, will be disturbed in the presence of gravitational fields, i.e. it will no longer be Galilean. In other words, geometric space is curved by the presence of gravitating bodies.

The gravitational fields of the Earth and sun cause continual rotation /18 of the line of apsides of the satellite orbit in the direction of its motion. This effect is proportional to the ratio V^2/C^2 (V is the velocity of the satellite in the gravitational field, C is the speed of light in vacuum). In addition to motion of the line of apsides, the gravitational effect of the sun also causes a shift in the line of nodes [21, 22].

According to [21], the shift in the perigee of the orbit of a heavenly body (in seconds of arc per century) due to the gravitational effect of the Earth may be determined from the formula

$$\Delta\omega_1 = 1{,}74 \cdot 10^{25}/a^{5/2}(1-e^2)$$

(a is measured in centimeters). The shift in the perigee due to the gravitational effect of the sun ($\Delta\omega_2$) for nearby objects in space is equal on the average to 1''.9 per century.

The shift in the line of nodes (in seconds of arc per century) due to the gravitational effect of the sun may be determined from the formula

$$\Delta\Omega = 1{,}67 \cdot 10^{33}/a^{5/2}(1-e^2).$$

Shown in Table 1 below [21] is the shift in the line of apsides of artificial satellite orbits (in seconds of arc per year) as a function of their distance from the center of the Earth (effect of the gravitational field of the Earth alone).

TABLE 1

Average Distance to Center of the Earth, cm	Orbital Eccentricity, e_0	Shift in the Line of Apsides, $\Delta\omega_1$
$r = r_{\text{Earth}} = 6{,}367 \cdot 10^8$	0	17''
$r = r_{\text{Earth}} + 4 \cdot 10^7 = 6{,}77 \cdot 10^8$	0,01	14'',5
$r = 10 \cdot 10^8$	0,25	5'',866

Tr. note: Commas indicate decimal points.

A second relativistic effect in the motion of satellites is displacement of the line of apsides caused by the rotation of gravitating bodies (the Earth and sun)[1]. The effect of rotation of the Earth and sun may be one or two orders of magnitude less than the relativistic effect which takes place in the absence of rotation. Both these effects add up algebraically, and accounting for the rotation of Earth and sun reduces the overall relativistic effect.

The equations of motion for artificial Earth satellites may be derived within the framework of relativistic mechanics. However, these equations cannot be justifiably used at present in view of their complexity and the comparatively slight deviation in the motion which they describe as compared /19
with motion in Galilean space. A. F. Bogorodskiy in particular [23] has derived equations of motion for artificial satellites with regard to the relativistic effects due to the uniformly rotating, homogeneous, sphereical Earth. In this same paper he determined the relativistic secular variations in the orbital elements of equatorial and polar circular satellites. According to his results, the drift in the line of apsides for an equatorial satellite is independent of the Earth's rotation, while the drift in the line of nodes for a polar satellite (extremely low in value, equal to only a few tenths of a second of arc per year) depends *only* on the Earth's rotation.

The figures given above show that the effect of the magnetic field and the relativistic effect are extremely small and may be disregarded in most problems on the motion of artificial Earth satellites.

Effect of Solar Radiation

The solar radiation factor, i.e. the pressure from solar rays, has an appreciable disturbing effect on the motion of space vehicles in which the ratio of cross-sectional area S to weight G is sufficiently large, *viz:*

$$S/G > 25 \text{ cm}^2/\text{ gram}.$$

A sufficiently strict computation of this effect is complicated by the fact that the vehicle spends part of the time in the Earth's shadow. Calculations are still more complicated for satellites which move in elliptical orbits [24, 25, 26].

Disturbances due to solar radiation in the case where the satellite is illuminated by the sun or completely eclipsed by the Earth have been studied specifically in [27]. The disturbing force in this instance (if we disregard parallax, which is equal to 11' at a distance of 1,600 km from the Earth) is equal to

[1] According to the general theory of relativity, the rotation of a gravitating body affects its gravitational field. /18

$$\vec{F} = F\vec{r}_S^{\circ},$$

where $\vec{r}_S^{\circ}$ is the basis vector for direction toward the sun; $F = \gamma\nu pS$; S is the effective cross-section of the satellite with respect to the direction of $\vec{r}_S^{\circ}$; p is the pressure of solar radiation close to the Earth (this quantity is equal to approximately $(4.5\text{-}10)\cdot 10^{-5}$ dyne/cm^2); γ is a coefficient which /20 depends on the reflecting properties of the satellite surface; ν is a coefficient which is equal to unity if the satellite is illuminated by the sun, and equal to zero if it is in the Earth's shadow.

Disturbances in the satellite trajectory due to solar radiation are given quite detailed consideration [28], where the effect of eclipsing of the satellite is also discussed as well as resonance problems (see below).

The pressure of solar rays on an artificial satellite moving in a circular orbit leads to a displacement of the center of the orbit (see for instance [29].). As a result, the distance between satellite and Earth is reduced in that part of its trajectory where it is moving away from the sun.

By way of example, we could point out that if the sun had been in the orbital plane of satellite 1960 ι_1 "Echo-1" (orbital parameters: $h_A = 1{,}750$ km, $h_\Pi = 1{,}633$ km, the period of the revolution $T = 121^m.6$, $S/G = 125$ cm^2/g immediately after placement in orbit), then the geometric center of the orbit would have been shifted by approximately 7 km per day. Actually ($i_0 \approx 47°$), the rate of the shift was 2-3 km/day according to data [30].

By comparing the disturbances due to the effect of solar radiation and the Earth's atmosphere, it was found [27] that the first effect predominates at altitudes of more than 900 km, while the second is predominant at lower altitudes.

At certain inclinations (depending on the radius of the orbit), the pressure of solar rays may induce resonance phenomena (see for instance [24], [28, 29]). In this case, the orbital elements vary in such a way that the perigee is always turned toward the sun. The altitude of the orbit's perigee is reduced by resonance effects, and the lifetime of an artificial satellite (before entering the denser layers of the atmosphere) may be reduced by a factor measured in the dozens. For the "Echo 1" satellite in particular, this "resonance" inclination is equal to 30°.

Thus, the phenomenon of solar radiation should be taken into consideration in studying the motion of satellites with a ratio $S/G > 25$ cm^2/gram over a long time interval[1].

The eccentricity of the Earth's gravitational field has a considerably stronger disturbing effect than the factors considered above. The following sections are devoted to an investigation of this effect on the motion of artificial Earth satellites. /21

§2. The Earth's Gravitational Field

Let us assume that a point mass m is located at the point (x_0, y_0, z_0) of an inertial rectangular coordinate system (x, y, z) with basis vectors $(\vec{i}, \vec{j}, \vec{k})$. According to the law of universal gravitation, the force with which this mass acts on a unit mass at a distance $r = \sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2}$, is equal to

$$\vec{F} = -f\frac{m}{r^2}\frac{\vec{r}}{r}, \tag{2.1}$$

where f is the gravitational constant (the constant of gravity or attraction) whose value depends on the chosen system of units (see Introduction).

In addition to (2.1), let us consider the function

$$V = fm/r. \tag{2.2}$$

Taking the gradient of this function, we get

$$\operatorname{grad} V = \vec{i}\frac{\partial V}{\partial x} + \vec{j}\frac{\partial V}{\partial y} + \vec{k}\frac{\partial V}{\partial z} =$$
$$= -\left(f\frac{m}{r^2}\frac{x-x_0}{r}\right)\vec{i} - \left(f\frac{m}{r^2}\frac{y-y_0}{r}\right)\vec{j} - \left(f\frac{m}{r^2}\frac{z-z_0}{r}\right)\vec{k}.$$

i.e.

$$\operatorname{grad} V = \vec{F}, \tag{2.3}$$

---

[1] Apparently the deviation in the actual position of the perigee for the "Vanguard 1" from the theoretical data (the parameters of its orbit are given above, $S/G \approx 0.14$ cm²/gram) is explained by the fact that the pressure of solar rays was not taken into consideration in the computations. /20

where $\vec{F}$ is a force which is defined in the given case by equation (2.1).

The function V which satisfies equation (2.3) is called a force function. We shall also call it the potential of force $\vec{F}$ or the potential of force field $\vec{F}$, without making any distinction between these terms. The attractive potential, which is inversely proportional to the distance between the interacting bodies, is called the Newtonian potential. It follows from the very definition of potential (2.3) that it has the dimension of energy (work).

If the point mass in coordinate system $(\vec{i}, \vec{j}, \vec{k})$ is replaced by a material body of mass M and volume T, then, if we represent it as consisting of n point masses and take the limit as $n \to \infty$, we get /22

$$V = f\int_M \frac{dm}{r} = f\int_T \rho\frac{d\tau}{r} = f\int_{x_m}\int_{y_m}\int_{z_m} \rho(x_m, y_m, z_m)\frac{dx_m\,dy_m\,dz_m}{r}, \tag{2.4}$$

where $r = \sqrt{(x-x_m)^2+(y-y_m)^2+(z-z_m)^2}$, and $d\tau = dx_m dy_m dz_m$.

Expression (2.4) is the gravitational potential of mass M on the external point (x, y,z) of unit mass. The quantity ρ is the probability density function of the mass, and integration is extended to the entire volume of the attracting body.

The components of the force of attraction acting on the external point may be found by direct differentiation of integral (2.4) with respect to coordinates x, y, z:

$$\left.\begin{aligned} F_x &= \frac{\partial V}{\partial x} = -f\int_{x_m}\int_{y_m}\int_{z_m}\rho\frac{x-x_m}{r^3}\,dx_m dy_m dz_m;\\ F_y &= \frac{\partial V}{\partial y} = -f\int_{x_m}\int_{y_m}\int_{z_m}\rho\frac{y-y_m}{r^3}\,dx_m dy_m dz_m;\\ F_z &= \frac{\partial V}{\partial z} = -f\int_{x_m}\int_{y_m}\int_{z_m}\rho\frac{z-z_m}{r^3}\,dx_m dy_m dz_m. \end{aligned}\right\} \tag{2.5}$$

Integrals (2.4) and (2.5) are taken in elementary functions only in some of the simplest cases. For instance, assuming that ρ = const and switching to spherical coordinates in (2.4), we may immediately (in view of spherical symmetry) derive equation (2.2) for the potential of a uniform sphere. Thus, the attractive potential for a uniform sphere is equal to the potential of a point with a mass equal to that of the body and located at the center of the sphere. When the sphere is separated from the gravitating body by an infinite distance, its potential (regardless of the shape of the body or the probability density law) approaches potential (2.2) as nearly as desired (the proof may be found [31]).

This fact facilitates the solution of many problems in celestial mechanics. However, in studying the motion of artificial Earth satellites, as well as the motion of space vehicles close to the Earth and other planets, it is necessary to consider the shape and dimensions of the attracting body. Since planets are not spherical in shape, and the density differs from various points in the body, i.e. $\rho \neq$ const, integral (2.4) may be only approximately computed, e.g. by representation in the form of an infinite series. /23

An expansion of this type with respect to spherical functions is written in the form

$$V(r,\vartheta,\lambda)=\frac{\mu}{r}\sum_{n=0}^{\infty}\sum_{m=0}^{n}\left(\frac{r_0}{r}\right)^n(c_{nm}\cos m\lambda+d_{nm}\sin m\lambda)P_{nm}(\cos\vartheta). \tag{2.6}$$

Expression (2.6) may be derived by various methods. One of these derivations is given in Appendix I.

The quantities r, ϑ and λ in this expression are the geocentric spherical coordinates of the external point acted upon by potential (2.6). The spherical latitude ϑ is reckoned from the axis $\vec{k}$ (which coincides with the axis of rotation of the Earth), and varies over the range $0 \leq \vartheta \leq \pi$[1]; the longitude λ is reckoned from the Greenwich meridian eastward in the region $0 \leq \lambda \leq 2\pi$. The constant $\mu = fM$, where M is the mass of the Earth. The quantity r_0 is taken as equal to the greatest equatorial radius of the Earth. The constants c_{nm} and d_{nm} are the coefficients of this expansion. The functions $P_{nm}(\cos\vartheta)$ are Legendre's polynomials, $P_{n0}(\cos\vartheta)$ (in some cases the second subscript may be omitted for simplicity) being called the principal Legendre's polynomial, $P_{nm}(\cos\vartheta)$ (when $m \neq 0$) is an adjoint Legendre function[2]; $P_{nm}(\cos\vartheta)\sin m\lambda$ and $P_{nm}(\cos\vartheta)\cos m\lambda$ are the elementary harmonics. The solution $Y(\vartheta,\lambda) = \Phi(\vartheta)\Lambda(\lambda)$ of equation (1.7) (see Appendix I) is called a spherical polynomial or a spherical function (sometimes also a space function)

[1] This system should be distinguished from the spherical astronomical coordinates to be discussed later in which the latitude ϑ is reckoned from the terrestrial equator.

[2] Also called associated Legendre functions of order n and index m. Expression (I.17') (Appendix I) shows the part played by the principal Legendre's polynomials: they determine all associated functions of a given degree.

$$Y_{nm} = (c_{nm} \cos m\lambda + d_{nm} \sin m\lambda) P_{nm}(\cos \vartheta),$$

while the function

$$V(r, \vartheta, \lambda) = R(r) Y(\vartheta, \lambda),$$

(see I.5), which is a solution for equation (I.2), is called a globular or solid function. Globular functions may be defined in the same way as rational integral functions (with respect to rectangular coordinates) of the n-th degree which satisfy Laplace's equation. /24

In order to understand the physical meaning of the individual terms in expression (2.6), let us examine the zeros of spherical functions.

When m = 0, we get the function $Y_{n0} = P_{n0} (\cos \vartheta)$ This function depends only on the spherical latitude ϑ and (as a polynomial of degree n) has n zeros in the interval $[0,\pi]$, (and as many zeros in the interval $[-\pi/2, \pi/2]$). Thus, the function $Y_{n0} = P_{n0} (\cos \vartheta)$, which is called a zonal function or zonal harmonic, vanishes at certain latitudes, forming n + 1 zones on the sphere within which it maintains its sign. Naturally, zonal functions can describe only the latitudinal effects of the gravitational field and shape of the Earth.

The spherical functions Y_{nm} when $m \neq 0$ contain both latitudinal and longitudinal terms

$$Y_{nm} = P_{nm}(\cos \vartheta) \begin{cases} \cos m\lambda, & n = 0, 1, 2, \ldots; \\ \sin m\lambda, & 0 < m \leq n. \end{cases}$$

If m = n, then according to (1.17') and (1.17) after a transition from x to cos ϑ, we get

$$P_{nm}(\cos \vartheta) = \frac{1}{2^n n!} \sin^n \vartheta \frac{d^{2n}}{[d \cos \vartheta]^{2n}} (\cos^2 \vartheta - 1)^n = \frac{(2n)!}{2^n n!} \sin^n \vartheta.$$

Then

$$Y_{nm} = \frac{(2n)!}{2^n n!} \sin^n \vartheta \begin{cases} \cos n\lambda; \\ \sin n\lambda. \end{cases}$$

These functions may vanish (with the exception of the poles, of course) only at the meridians defined by the equations

$$\cos n\lambda = 0 \text{ and } \sin n\lambda = 0.$$

They are called sectoral spherical functions, and they describe the purely longitudinal effects of the gravitational potential.

In the case where $m \neq n$, the functions must take on zero values along n parallels and m meridians. Therefore, on the surface of the sphere, they maintain their sign within curvilinear quadrilaterals and triangles formed by the intersection of two parallels and two meridians, or two meridians and a parallel, and are called tesseral harmonics (Latin *tessera*, a plate). /25
Tesseral harmonics reflect the mixed effects of the gravitational field (effects which depend on both latitude and longitude). The number of zeros for all harmonics increases with an increase in n or m. Consequently, the general governing principals with respect to the shape and potential of the Earth are described by functions of a lower degree. Local changes, on the other hand, are associated with functions of a higher degree. In this regard, the amplitudes of harmonics tend to decrease with an increase in the degree of the function.

The coefficients of expansion (2.6) may be given a physical interpretation: they are expressed in terms of integrals of the form

$$I_{\alpha\beta\gamma} = \frac{1}{Mr_0^n} \int_M x^\alpha y^\beta z^\gamma \, dm, \quad \alpha+\beta+\gamma=n \quad (\alpha, \beta, \gamma \text{ -- are whole numbers}) \tag{2.7}$$

and linear combinations of these integrals [32], and the quantities $I_{\alpha\beta\gamma}$ when $n \leq 2$ are proportional to the coordinates of the center of inertia (center of mass) and the moments of inertia. When $n > 2$, the physical meaning of integrals (2.7) becomes less clear. In Appendix III, where this problem is considered in more detail , the discussion is limited to values of n less than or equal to 2, although theoretically, similar relationships may also be found for $n > 2$. For instance, in [31 and 32] the coefficients in expansion (2.6) are expressed in terms of relationships (2.7) up to a value of $n = 4$. According to the results of Appendix III, we may write

$$\left.\begin{array}{l} c_{00}=1;\ \ c_{10}=Z_c/r_0;\ \ c_{11}=X_c/r_0;\ \ d_{11}=Y_c/r_0;\\ c_{20}=\dfrac{A+B-2C}{2Mr_0^2};\ \ c_{21}=\dfrac{E}{Mr_0^2};\ \ d_{21}=\dfrac{D}{Mr_0^2};\\ \qquad\qquad c_{22}=\dfrac{B-A}{4Mr_0^2};\ \ d_{22}=\dfrac{F}{2Mr_0^2}. \end{array}\right\} \tag{2.7'}$$

Here M is mass; r_0 is the mean radius of the Earth (heretofore the greatest equatorial radius was taken as r_0; there is no contradiction here since any quantity close to the radius of the Earth may be taken); X_c, Y_c, Z_c are the coordinates of the center of gravity or inertia (see (III.9)); A, B, C are the moments of inertia of the Earth with respect to the principal axes (see (III.12)); and D, E, F are the centrifugal moments of inertia (see (III.12')).

Let us note that if the body is dynamically symmetric with respect to axis Z_1 (see Appendix III), then B = A. In this case, $c_{21} = d_{21} = c_{22} = d_{22} = 0$, and $c_{20} = (C - A)/Mr_0^2$. The coefficient c_{20} is of the same order as the quantity (C - A)/C, which is called the dynamic flattening of the Earth.

Since the moments of inertia are proportional to the product of the mass by the square of a linear dimension of the body, quantities (2.7') are dimensionless. The same may be said of the remaining coefficients of expansion (2.6), since they are all expressed only in terms of dimensionless integrals of the form $I_{\alpha\beta\gamma}$. /26

If the origin of the coordinate system coincides with the center of gravity of the Earth, then $c_{10} = c_{11} = d_{11} = 0$. If in addition, the coordinate axes coincide with the axes of inertia, then all centrifugal moments of inertia will be equal to zero, and consequently, $c_{21} = d_{21} = 0$[1].

Theoretically, proper selection of the initial meridian would also make coefficient d_{22} equal to zero. However, Greenwich is taken as the origin for readings, and d_{22} has a non-zero value.

---

[1] According to Ref. [32], $\left|\dfrac{\sqrt{c_{21}^2+d_{21}^2}}{c_{20}}\right| = \theta < 0.5\cdot 10^{-5}$, where θ is the angle between the axis of rotation of the Earth (axis $\vec{k}$) and the closest principal axis of inertia.

Since a geocentric coordinate system (rectangular or spherical) with just such an arrangement of axes is selected as a rule in studying the motion of artificial Earth satellites, there are no harmonics of the first degree or tesseral harmonics of the second degree in expansion (2.6).

Taking consideration of everything said up to this point, we get instead of expansion (2.6)

$$V(r, \vartheta, \lambda) = \frac{\mu}{r}\left\{1 + \left(\frac{r_0}{r}\right)^2 [c_{20} P_{20}(\cos\vartheta) + (c_{22}\cos 2\lambda + d_{22}\sin 2\lambda)P_{22}(\cos\vartheta)] + \sum_{n=3}^{\infty}\sum_{m=0}^{n}\left(\frac{r_0}{r}\right)^n (c_{nm}\cos m\lambda + d_{nm}\sin m\lambda)P_{nm}(\cos\vartheta)\right\}. \quad (2.6')$$

However, in the following discussion, for the sake of brevity we shall write out the potential of attraction in form (2.6) without losing sight of the fact that it has form (2.6').

In solving problems relating to the motion of artificial Earth satellites, it is convenient to use the complement of ϑ rather than ϑ itself. This angle $\varphi = \frac{\pi}{2} - \vartheta$ is reckoned from the equatorial plane of the Earth (which coincides with plane $\vec{i} \times \vec{j}$) northward ($0 < \varphi \leq \pi/2\varphi$) and southward ($-\pi/2 \leq \varphi < 0$). It is equal to geographic latitude on the Earth's surface, or to declination in the equatorial system of astronomic coordinates.

In this case, (2.6) will take the form

$$V(r,\varphi,\lambda) = \frac{\mu}{r}\sum_{n=0}^{\infty}\sum_{m=0}^{n}\left(\frac{r_0}{r}\right)^n (c_{nm}\cos m\lambda + d_{nm}\sin m\lambda)P_{nm}(\sin\varphi). \quad (2.8)$$

Sometimes equation (2.6) is completely written out in astronomical equatorial coordinates r, φ, t_{Gr} (see for instance [32]) where t_{Gr} is the Greenwich angle reckoned westward, $t_{Gr} = 360° - \lambda$. Then for V(r, φ, t_{Gr}) we get the expression

$$V(r, \varphi, t_{Gr}) = \frac{\mu}{r}\sum_{n=0}^{\infty}\sum_{m=0}^{n}\left(\frac{r_0}{r}\right)^n (c_{nm}\cos m t_{Gr} + d_{nm}\sin m t_{rp})P_{nm}(\sin\varphi). \quad (2.9)$$

By using the formulas

$$\left.\begin{array}{l} \sin\varphi = z/r; \quad \cos\varphi = \sqrt{x^2+y^2}/r; \\ \sin\lambda = y/\sqrt{x^2+y^2}; \quad \cos\lambda = x/\sqrt{x^2+y^2}; \\ r = \sqrt{x^2+y^2+z^2}, \end{array}\right\} \tag{2.10}$$

we may convert equation (2.6) to rectangular coordinates.

Expressions (2.6), (2.8) and (2.9) correspond to the notation derived by I. D. Zhongolovich for the attractive potential of the Earth [32], which is extensively used both in the Soviet Union and abroad. In this regard, the international notation used for the coefficients (for the expression written in form (2.8)) is $c_{nm} = C_{nm}$, $d_{nm} = S_{nm}$.

In the British and American literature, the gravitational potential is frequently written as the potential of an ellipsoid of revolution, represented by the formula

$$V = f\frac{M}{r}\left[1 - \sum_{n=2}^{\infty} I_n \left(\frac{r_0}{r}\right)^n P_n(\varphi)\right] \tag{2.11}$$

or Jeffreys [33]

$$V = f\frac{M}{r_0}\left[\frac{r_0}{r} + I\frac{r_0^3}{r^3}\left(\frac{1}{3} - \sin^2\varphi\right) + \frac{8}{35}D\frac{r_0^5}{r^5}P_{40}\right]. \tag{2.12}$$

The quantities I and D, which are called the Jeffreys constants, are related to c_{20} and c_{40} by the expressions

$$I = -\frac{3}{2}c_{20}; \quad D = \frac{35}{8}c_{40}. \tag{2.13}$$ /28

The coefficients of the expansion for the gravitational potential are determined from measurements of the Earth's gravitational force field.

§3, Gravitational Force Field

In cases where the motion of a body is being considered in a coordinate system tied to the rotating Earth, the body is subjected to a translational centrifugal force. The resultant of this force and the Earth's attraction is called the force of gravity.

The centrifugal force is perpendicular to the axis of rotation and has the components (in the absolute coordinate system) $(-\omega^2 x, -\omega^2 y, 0)$.

Thus, the modulus of this force is

$$P = \omega^2 r \cos\varphi , \tag{3.1}$$

where ω is the angular velocity of the Earth's rotation; r is the distance from the center of gravity of the Earth to the moving point which has geographic latitude φ (reckoned from the equatorial plane).

It is assumed in this case that the Earth rotates as an absolutely rigid body with a constant vector of angular velocity. The rate of rotation of the Earth is quite accurately known at present, and is equal to $\omega = 7.29211 \cdot 10^{-5}$ sec [1].

The force P has the potential function

$$U = \frac{1}{2}\,\omega^2 r^2 \cos^2\varphi \;^{1)}. \qquad (3.2)^1$$

The expression for the potential of the force of gravity W may be written in the form

$$W = V + U; \tag{3.3}$$

U is given by relationship (3.2), and V is the potential of attraction determined according to §2.

---

[1] Actually, function (3.2) satisfies the condition

$$P = |\operatorname{grad} U| = \sqrt{\left(\frac{\partial u}{\partial r}\right)^2 + \left(\frac{1}{r}\frac{\partial u}{\partial \varphi}\right)^2} = \omega^2\sqrt{r^2\cos^4\varphi + r^2\sin^2\varphi\cos^2\varphi} = \omega^2 r\cos\varphi .$$

However, U is not a harmonic function since $\Delta U = 2\omega^2$.

Surfaces where the potential maintains a constant value are called equipotential surfaces or level surfaces. For the potential W, such surfaces

$$W = V + U = \text{const} \tag{3.4}$$

have recently become known as geops.

The direction of the force of gravity is everywhere normal to equipotential surfaces as otherwise the tangential component of the force of gravity would cause the body to move along the surface (3.4).

In point of fact, if the effect of other heavenly bodies is disregarded as well as some other insignificant effects, the particles of a fluid surface would be subjected to the force of gravity alone, which in this case would determine the shape of the attracting body. However, the surface of the Earth is only partially covered by ocean, and the terrain of the dry land is complex. Therefore, the shape of the Earth is described by introducing some arbitrary surface which differs from the true physical surface and is called the geoid (this term was introduced by Listing in 1873).

The geoid is defined as the mean free surface which would be shown by a worldwide ocean extending beneath islands and continents. It is assumed that the mass of the continents and islands in this case is condensed directly beneath the surface of this arbitrary ocean. When we say "mean free surface", we assume that there are no tidal effects (i.e. we disregard not only the effect of the moon, but also that of other heavenly bodies) and that there are no disturbances due to winds. It follows from what we have said that the surface of the geoid is the equipotential surface of the Earth's gravity field.

The concept of the geoid is not sufficiently strict. The geoid may be thought of as the surface of the ocean filling channels cut beneath the continents and continuing the free surface of the water. The geoid may also be represented in another way as the analytical continuation of the ocean's surface beneath the continents. H. Poincaré [34] showed that these surfaces do not coincide[1]. The distance between them is proportional to the square /30
of the altitude of the Earth's surface above the geoid.

It follows from expression (3.4) that the shape of the geoid (as well as the gravitational potential) may be described by an infinite series of

[1] As a rule, the first of the two definitions given above is adhered to in /29
the literature. The use of the geoid concept for describing the shape and potential field of the Earth is traditional, but not natural, and perhaps not the most convenient. M. S. Molodenskiy's theory, which is based on other principles, is briefly outlined below.

spherical functions. The force of gravity g (acting on a unit mass) or more precisely the acceleration due to gravity, is expressed in the same way:

$$g = \sqrt{\left(\frac{\partial W}{\partial r}\right)^2 + \left(\frac{1}{r}\frac{\partial W}{\partial \psi}\right)^2 + \left(\frac{1}{r\cos\psi}\frac{\partial W}{\partial \lambda}\right)^2}\,. \tag{3.5}$$

For g we may write the series

$$g = \sum_{n=0}^{\infty}\sum_{m=0}^{n}\left(\frac{r}{r_0}\right)^n (A_{nm}\cos m\lambda + B_{nm}\sin m\lambda) P_{nm}(\sin\varphi)\,. \tag{3.6}$$

Just as in expression (2.6), the coefficients A_{10}, A_{11}, B_{11}, A_{21} and B_{21} should be set equal to zero.

In many cases it is necessary to go from expansion (3.5) to an expansion of the gravitational potential of form (2.6). I. D. Zhongolovich considers a transition of this type [35] resulting from a comparison of expressions (2.6'), (3.3), (3.5) and (3.6).

In the case where the values of c_{nm} and d_{nm} are known, the coefficients A_{nm} and B_{nm} when $n \leq 4$ are found from the equations

$$\left.\begin{aligned}
A_{00} &= g_0\Big(1 - \tfrac{8}{3}m - \tfrac{68}{15}m^2 + \tfrac{44}{15}mc_{20} + \\
&\qquad + \tfrac{4}{5}c_{20}^2 + \tfrac{48}{5}c_{22}^2 + \tfrac{48}{5}d_{22}^2\Big);\\
A_{20} &= g_0\Big(c_{20} + \tfrac{8}{3}m + \tfrac{52}{21}m^2 - \tfrac{76}{21}mc_{20} - \\
&\qquad - \tfrac{1}{7}c_{20}^2 + \tfrac{12}{7}c_{22}^2 + \tfrac{12}{7}d_{22}^2\Big);\\
A_{22} &= g_0\Big(c_{22} - \tfrac{68}{7}mc_{22} + \tfrac{2}{7}c_{20}c_{22}\Big);\\
B_{22} &= g_0\Big(d_{22} - \tfrac{68}{7}md_{22} + \tfrac{2}{7}c_{20}d_{22}\Big);\\
A_{30} &= g_0 2c_{30};\\
A_{31} &= g_0 2c_{31};\\
B_{31} &= g_0 2d_{31};\\
A_{32} &= g_0 2c_{32};\\
B_{32} &= g_0 2d_{32};
\end{aligned}\right\} \tag{3.7}$$

$$
\left.\begin{aligned}
A_{33} &= g_0 2c_{33};\\
B_{33} &= g_0 2d_{33};\\
A_{40} &= g_0\Big(3c_{40} + \tfrac{72}{35}m^2 + \tfrac{24}{35}mc_{20} - \tfrac{198}{35}c_{20}^2 - \\
&\qquad\qquad - \tfrac{396}{35}c_{22}^2 - \tfrac{396}{35}d_{22}^2\Big);\\
A_{41} &= g_0 3c_{41};\\
B_{41} &= g_0 3d_{41};\\
A_{42} &= g_0\Big(3d_{42} + \tfrac{4}{35}mc_{22} - \tfrac{66}{35}c_{20}c_{22}\Big);\\
B_{42} &= g_0\Big(3d_{42} + \tfrac{4}{35}md_{22} - \tfrac{66}{35}c_{20}d_{22}\Big);\\
A_{43} &= g_0 3c_{43};\\
B_{43} &= g_0 3d_{43};\\
A_{44} &= g_0\Big(3c_{44} - \tfrac{33}{70}c_{22}^2 + \tfrac{33}{70}d_{22}^2\Big);\\
B_{44} &= g_0\Big(3d_{44} - \tfrac{33}{35}c_{20}d_{22}\Big);\\
m &= \omega^2 r_0/(2g_0);\\
g_0 &= fM/r_0^2 = \mu/r_0^2.
\end{aligned}\right\} \qquad (3.7)
$$

It should be pointed out that the quantities mc_{22}, md_{22}, $c_{20}c_{22}$ and $c_{20}d_{22}$ are of the third negative order of magnitude and are comparable with such products (not included in formulas (3.7)) as $c_{20}c_{30}$ and $c_{20}c_{40}$, while terms containing c_{22}^2 and d_{22}^2 are generally of the fourth negative order of magnitude. The reason that these quantities should be taken into account in equations (3.7) will be apparent later (§5) when we consider a model for the gravitational field of the triaxial Earth where the harmonics corresponding to coefficients c_{22} and d_{22} are the most significant terms describing this effect which appear in the expansion for the potential. /31

When going the other way (expressing coefficients c_{nm} and d_{nm} in terms of A_{nm} and B_{nm}), the quantities τ_{nm} and π_{nm} should first be calculated (for all values of n and m) by the formulas

$$\tau_{nm} = A_{nm}/A_{00}; \quad \pi_{nm} = B_{nm}/A_{00}.$$

The coefficients of the expansion for the attractive potential are then determined (for n $\leq$ 4) from the equations:

$$\left.\begin{aligned}
c_{20} &= \tau_{20} - \frac{8}{3} m - \frac{100}{9} m^2 + \frac{4}{21} m \tau_{20} + \\
&\qquad + \frac{1}{7} \tau_{20}^2 - \frac{12}{7} (\tau_{22}^2 + \pi_{22}^2); \\
c_{22} &= \tau_{22}\left(1 - \frac{2}{7}\tau_{20} + \frac{164}{21} m\right); \\
d_{22} &= \pi_{22}\left(1 - \frac{2}{7}\tau_{20} + \frac{164}{21} m\right); \\
c_{30} &= \frac{1}{2} \tau_{30}; \\
c_{31} &= \frac{1}{2} \tau_{31}; \\
d_{31} &= \frac{1}{2} \pi_{31}; \\
c_{32} &= \frac{1}{2} \tau_{32}; \\
d_{32} &= \frac{1}{2} \pi_{32}; \\
c_{33} &= \frac{1}{2} \tau_{33}; \\
d_{33} &= \frac{1}{2} \pi_{33}; \\
c_{40} &= \frac{1}{3} \tau_{40} + \frac{40}{3} m^2 - \frac{72}{7} m \tau_{20} + \frac{66}{35} \tau_{20}^2 + \frac{132}{35} (\tau_{22}^2 + \pi_{22}^2); \\
c_{41} &= \frac{1}{3} \tau_{41}; \\
d_{41} &= \frac{1}{3} \pi_{41}; \\
c_{42} &= \frac{1}{3} \tau_{42} - \frac{12}{7} m \tau_{22} + \frac{22}{35} \tau_{20} \tau_{22}; \\
d_{42} &= \frac{1}{3} \pi_{42} - \frac{12}{7} m \pi_{22} + \frac{22}{35} \tau_{20} \pi_{22}; \\
c_{43} &= \frac{1}{3} \tau_{43}; \\
d_{43} &= \frac{1}{3} \pi_{43}; \\
c_{44} &= \frac{1}{3} \tau_{44} + \frac{11}{70} (\tau_{22}^2 - \pi_{22}^2); \\
d_{44} &= \frac{1}{3} \pi_{44} + \frac{11}{35} \tau_{22} \pi_{22}.
\end{aligned}\right\} \qquad (3.8)$$

The discussion above concerning inclusion of quantities of the third and fourth negative orders of magnitude in formulas (3.7) applies to formulas (3.8) as well. /33

The quantity r_0 is defined as the mean radius of the Earth or as the greatest equatorial radius.

In equations (3.8), $m = \frac{\omega^2}{2}\frac{b}{A_{00}}(1-\tau_{20})$, , where b is the semiminor axis of the terrestrial ellipsoid.

The acceleration due to gravity g is frequently given in the form of an expansion with respect to normalized spherical functions $\bar{P}_{nm}$ determined from the equation

$$\bar{P}_{nm}(\varphi,\lambda) = \sqrt{(2n+1)\delta\frac{(n-m)!}{(n+m)!}}\,P_{nm}(\varphi)\,\frac{\cos m\lambda}{\sin m\lambda},$$
$$\delta = 1 \text{ when } m = 0, \quad \delta = 2 \text{ when } m > 0. \tag{3.9}$$

The normalized spherical function is distinguished by the fact that its mean-square value on the sphere is equal to unity. In expanding the force of gravity with respect to functions $\bar{P}_{nm}$, with coefficients $\bar{A}_{nm}$, $\bar{B}_{nm}$, the relationship between them and the quantities A_{nm}, B_{nm} (see (3.6)) is given by the equations

$$\left.\begin{aligned}\bar{A}_{nm} &= \sqrt{\frac{1}{(2n+1)\delta}\frac{(n+m)!}{(n-m)!}}\,A_{nm};\\ \bar{B}_{nm} &= \sqrt{\frac{1}{(2n+1)\delta}\frac{(n+m)!}{(n-m)!}}\,B_{nm}.\end{aligned}\right\} \tag{3.9'}$$

Expressions like these also give the relationship between normalized and non-normalized coefficients c_{nm} and d_{nm} in the expansion for the gravitational potential.

Equations (3.9) and (3.9') may always be used for finding the coefficients A_{nm} and B_{nm} in the expansion for the gravitational potential from the known values of the coefficients in the expansion for the potential of the force of gravity. Several authors have carried out computations of this type (on the basis of various initial material). This was done in the Soviet Union by I. D. Zhongolovich [32] in 1957 for the first eight spherical functions of the potential expansion. I. D. Zhongolovich's coefficients are given in Appendix IV.

In the case where the formula for gravitational potential in form (2.12) is used, the coefficients I and D are determined from (2.14). Jeffreys [33] gives the following values:

$$I = 0.0016370 \pm 0.0000041; \; D = 0.0000107$$

(when data from observations of artificial Earth satellites are taken into /34
consideration, $I = 0.001630 \pm 0.000001$).

The numerical values of the coefficients I_i in formula (2.12) are equal to the left of the decimal point to coefficients c_{n0}, and may be taken from Appendix IV. The values of I_i found by American authors are given in this same section.

During gravitational measurements, local irregularities in the mass distribution of the Earth may cause local variations in the force of gravity. Therefore, gravitational measurements must be distributed as uniformly as possible over the Earth's surface, and the number of measurements should be considerably greater than the number of coefficients to be determined.

The necessity for an excess number of known values of the force of gravity is also the result of representational errors (i.e., errors due to the discrete nature of gravimetric measurements), as well as partially unavoidable errors in measurement. The smoothed statistical values of the coefficients in the expansion are always determined in practice. This explains the fact that an increase in the number of coefficients to be determined (for a fixed number of measurements) leads to a reduction in their accuracy.

Since gravimetric measurements are made at different geographic points and at different altitudes with respect to sea level (i.e. on the so-called physical surface of the Earth, which is not an equipotential surface), the use of these measurements for determining the parameters of the gravitational field is a fairly complex problem.

One of the methods for solving this problem, the traditional method, boils down to reducing measurements of the force of gravity to the surface of the geoid. However, in this case, we are faced with the difficulty of plotting the geoid in regions where external masses are present, and of analytical continuation of the force of gravity through these masses.

Various methods may be used for transposing projecting masses to the interior of the geoid or onto its surface. This process is called regularization of the Earth, and the resultant figure is called the regularized geoid. However, the use of a regularized geoid does not eliminate the uncertainty due to ignorance of the actual distribution of masses between sea level and the physical surface.

The complex geometric shape of the geoid results in inconvenient mathematical relationships which make it difficult to analyze measurements of the values of g. Therefore, instead of the geoid, an ellipsoid called the reference ellipsoid or the geodetic ellipsoid is taken as the reference surface. /35 The choice of an ellipsoid as a reference surface is also explained by the possibility in this case for using the results of the Stokes theory (as well as Molodenskiy's theory, as we shall see later) which states that the potential of the force of gravity of a rotating mass is uniquely determined by giving the mass of the body, the external equipotential surface and the rate of rotation. If we use this procedure for determining the potential of a body (ellipsoid) sufficiently close to the shape of the Earth, then finding the external potential reduces to the determination of small differences whose squares may be disregarded. In most cases, the reference ellipsoid is an ellipsoid of revolution[1] with a shape characterized by the amount of flattening α

$$\alpha = (a - b)/a$$

where a and b are the axes of the ellipsoid.

The shape of the Earth, just as the potential, is found in the form of deviations of the geoid from the reference ellipsoid.

In connection with the use of the reference ellipsoid as the simplest surface, only slightly differing from the geoid, we introduce the concept of the normal force of gravity as the force of gravity on the chosen reference surface.

The normal force of gravity (designated in gravimetry by the letter γ) is expressed by a simple relationship which may be easily used for finding its value at any point on the ellipsoid surface. Because of this, the problem of studying the force of gravity itself is replaced by an investigation of comparatively small deviations in the force of gravity from the normal values (so-called anomalies). Reductions of the force of gravity are likewise replaced by the reductions of anomalies.

The difference between the complete and normal values of the potential is called the disturbing potential. The use of the disturbing potential makes it possible to limit computations to the linear theory in determining the coefficients of the expansion.

---

[1] A triaxial ellipsoid may also be taken (see for instance [36, 37, 38]). In this case its shape is determined by the amount of polar flattening α and equatorial flattening $\gamma = (b - c)/b$.

Several formulas are known for the normal force of gravity [36, 37]. Let us cite a few of them.

The Helmert Formula (1901 - 1909) /36

$$\gamma_0 = 978.030(1 + 0.005302 \sin^2\varphi - 0.000007 \sin^2 2\varphi)$$

derived for an ellipsoid with flattening α = 1:298.2.

Cassini's Formula (1930)

$$\gamma_0 = 978.049(1 + 0.0052884 \sin^2\varphi - 0.0000059 \sin^2 2\varphi)$$

derived for a Hayford spheroid with parameters α = 1:297.0, a = 6,378,388 m.

Zhongolovich's Formula (1952) for an ellipsoid of revolution

$$\gamma_0 = 978.0573(1 + 0.0052837 \sin^2\varphi - 0.0000059 \sin^2 2\varphi)$$

corresponds to an ellipsoid with parameters α = 1:296.6, a = 6,378,070 m.

Expressions for the normal force of gravity may also be written in spherical functions. In this case, the Helmert formula takes the form

$$\gamma_0 = 979754.85 + 3454.40 P_{20}(\sin\varphi) + 6.26 P_{40}(\sin\varphi)$$

or in normalized functions

$$\gamma_0 = 979754.85 + 1544.85 \bar{P}_{20}(\sin\varphi) + 2.09 \bar{P}_{40}(\sin\varphi).$$

The Hayford spheroid with the parameters cited above has been recommended as an international reference for geodetic projects. Geodetic measurements in the Soviet Union are based on the Krasovskiy ellipsoid (a = = 6,378,245 m, α = 1:298.3). It has recently come to light that this figure represents the entire Earth as a whole better than the Hayford spheroid.

The theoretical impossibility of determining the shape of the geoid has made it necessary to look for other methods of studying the gravitational field of the Earth. A new approach has been proposed by M. S. Molodenskiy, who has formulated and solved the problem of determining the external gravitational field for the general case where measurements are given on the physical (nonequipotential) surface which encloses all masses of the Earth.

The disturbing potential T may be found from a solution of the third boundary value problem of potential theory. The distinguishing characteristic of the Molodenskiy problem lies in the fact that the surface on which the limiting boundary condition is defined in the given case is only approximately known. This boundary surface, i.e. the surface of the Earth, is the one to be determined.

Molodenskiy reduced the solution of the given problem to solution of an integral equation with respect to the disturbing potential T or the altitude anomaly ζ. In this case, the physical surface of the Earth and the /37
external gravitational field are not regularized, and determination of the shape of the Earth reduces to finding geodetic altitudes (altitudes of the physical surface of the Earth above the reference ellipsoid) which may be strictly determined from measurements.

To solve the integral equation which he had derived, M. S. Molodenskiy expressed the disturbing potential in the form of the potential of a simple layer of density φ distributed over the surface S [39]:

$$T = \int_S \frac{\varphi}{r} ds.$$

In this case, the force of gravity in outer space is represented as the sum of the principal part (i.e. the normal force of gravity) and the attraction of layer φ located on the physical surface of the Earth. The value of density φ is found by solving Molodenskiy's integral equation which contains the values for the anomalies in the force of gravity found by measurements on the physical surface. Thus, measurements of the force of gravity may be used for direct determination of the parameters of the external gravitational field.

The greatest difficulties which arise during practical implementation of this procedure for solution of a problem are associated with the fact that the measurements of the force of gravity are not densely distributed over the entire physical surface, but rather are made at discrete points. The necessity therefore arises for representing this quantity in regions where no measurements have been made. Topographic maps may be used for calculating the most highly anomalous part of the variation in the force of gravity (that part which is due to the attraction of topographic masses) accurate to 10-15%. It is preferable to totally eliminate the effect of topographic masses (those projecting above the equipotential surface) from the anomalies in the force of gravity, but to take this factor into account later in derivation of the disturbing potential T (see L. P. Pellinen [40]). Mathematically, this is also achieved by isolating the effect of a spherical layer of thickness H_0 in the effect of the topographic masses g_p. The geometric interpretation [40] reduces to construction of a smoothed physical surface which passes through gravimetric and astronomogeodetic points. In this process, the projecting topographic masses are removed and depressions in the actual physical surface are filled in. This transposition of masses results in smoothing of the physical surface. Hence, Molodenskiy's theory will be applied to the remaining anomalies, and the physical surface will be /38 smoothed to such an extent that it will be possible to use the Stokes series (which relates the values of T and ζ to anomalies in the force of gravity and is theoretically valid if the physical surface coincides with the equipotential surface), i.e. to use a well-known procedure for finding the unknown quantities.

Gravimetric surveys take Potsdam (East Germany) as the international reference point, since the first precision measurements of the absolute force of gravity were made there. Relative measurements are made at all remaining reference points in each country, so that all measurements are made in a single system called the Potsdam system. The results of absolute measurements in a number of localities show that the absolute value determined at Potsdam is too high by approximately 13 mgal[1]. This correction must always be made in quantities derived in the Potsdam system.

Gravimetric measurements are not the only method for determining the figure of the Earth. Besides, in and of themselves, these measurements may only be used for determining the shape of the Earth. In contrast, the geometric method (which utilizes problems solved by higher geodesy) generally speaking, permits determination not only of the Earth's shape, but also its dimensions. The advantages of using gravimetric measurements rather than geodetic are related to the fact that these measurements may be made not only on dry land, but also on the ocean which covers the greater part of the Earth's surface. In general, however, these two methods complement each other: a strict solution of geodetic problems is impossible without using gravimetric data, and vice versa.

These procedures for determining the figure of the Earth (and hence its gravitational field) are not the only such techniques. The astronomical method based on studying perturbations in the motion of artificial satellites and the method of space triangulation have recently come into use for determining the shape and dimensions of the Earth. The investigation of satellite

[1] A milligal (mgal) is equal to 0.001 gal--the unit of acceleration in the CGS system (named in honor of Galileo). This is the acceleration imparted to a mass of 1 gram by a force of dyne. The total mean acceleration due to gravity on the surface of the Earth is 970.1 gals.

motion provides for considerable refinement in the parameters of the geoid. In this connection, artificial satellites are better for determining the coefficients of lower harmonics in the potential expansion, while gravimetric /39 methods are preferable for the coefficients of upper harmonics. In particular, the most recent determinations of polar flattening of the Earth, which utilized satellite data, are apparently the most accurate. For instance, J. Kozai (United States), using information on satellite motion for determining polar flattening, got $\alpha = 1:298.31 \pm 0.01$ [41], and H. Jeffreys got $\alpha = 1:298.05 \pm 0.11$ [33] (see also Appendix IV).

Fundamental research on the gravitational field of the Earth was begun in 1952 by I. D. Zhongolovich who used the gravimetric material available up to that time for determining the parameters of the Earth and the coefficients in the expansion for the force of gravity up to and including the eighth harmonic [35]. This work was used as a basis in [32] for determining the coefficients in the expansion for the attraction potential up to and including the fourth harmonic[1]. The polar flattening found in these computations is equal to $\alpha = 1:296.6$.

In recent years, Kaula (United States) has computed the coefficients in the expansion for the force of gravity up to and including the eighth harmonic [42]. He obtained for polar flattening and the semimajor axis of the general terrestrial ellipsoid: $\alpha = 1:298.24 \pm 0.01$, and $a = 6{,}378{,}163 \pm$ $\pm$ 15 meters, respectively.

Given for comparison in Table 2 below are the coefficients of the normalized spherical harmonics in the expansion of the potential for gravitational anomalies computed by Zhongolovich and Kaula for the normal Helmert formula. The table also gives the mean square errors (according to Kaula [42]) in determining the coefficients of the expansion.

It is evident from the table that the coefficients as determined by Kaula differ considerably from those determined by Zhongolovich. This is explained by the difference in the number of gravimetric data (a much greater amount of material was available to Kaula), their methods of analysis, and the hypotheses on which the procedures were based. Apparently Zhongolovich's values are somewhat too high, while Kaula's are too low.

As we have already mentioned previously, the coefficients in the expansion for gravitational potential found by I. D. Zhongolovich are given in Appendix IV. This same appendix also gives the values of the coefficients, according to non-Soviet sources.

[1] According to materials published [35], the coefficients may theoretically be found to the eight harmonic.

TABLE 2*

		A_{nm}, mgal		$\sigma\{A_{nm}\}$, mgal	B_{nm}, mgal		$\sigma\{B_{nm}\}$, mgal
n	m	Zhongolovich	Kaula	Kaula	Zhongolovich	Kaula	Kaula
0	0	+22,0	+13,2	–	–	–	–
1	0	– 0,1	0,0	–	–	–	–
1	1	+ 0,3	0,0	–	+0,9	0,0	–
2	0	– 5,1	+ 1,1	–	–	–	–
2	1	+ 0,11	0,0	–	+0,5	0,0	–
2	2	+ 7,6	+ 0,74	0,87	– 1,7	–0,4	0,93
3	0	+ 3,2	+ 1,52	0,20	–	–	–
3	1	+ 3,5	+ 2,02	0,86	–1,8	+0,76	0,84
3	2	+ 2,0	+ 1,90	0,96	+2,9	–0,06	0,98
3	3	+ 5,9	+ 1,12	0,89	+4,7	+2,75	0,87
4	0	+ 0,4	– 1,3	0,20	–	–	–
4	1	– 0,7	– 1,85	0,90	–0,9	–0,46	0,90
4	2	+ 0,1	+ 1,36	0,94	+1,3	+1,23	0,90
4	3	+ 1,00	+ 1,49	0,89	+0,2	–0,04	0,85
4	4	+ 0,9	– 0,30	0,84	+1,2	+1,06	0,85
5	0	+ 0,8	– 0,10	0,46	–	–	–
5	1	– 1,3	– 1,41	0,81	+0,2	+0,08	0,82
5	2	+ 0,5	+ 1,07	0,84	+0,2	–0,72	0,83
5	3	– 2,1	– 0,16	0,82	–0,1	+0,10	0,82
5	4	– 0,5	– 0,51	0,78	–0,6	–0,55	0,79
5	5	+ 0,3	– 0,34	0,76	+0,2	–1,34	0,79
6	0	+ 1,8	– 0,27	0,26	–	–	–
6	1	– 0,5	– 0,36	0,73	–0,2	+0,48	0,72
6	2	+ 0,2	+ 0,43	0,74	–0,80	–0,90	0,73
6	3	– 1,1	– 0,26	0,75	+1,0	+0,06	0,74
6	4	– 0,3	+ 0,11	0,73	–1,2	–1,36	0,71
6	5	– 1,4	– 1,28	0,69	–1,9	–2,38	0,70
6	6	+ 0,4	+ 0,29	0,69	+0,1	–0,77	0,70
7	0	+ 1,1	– 0,22	0,56	–	–	–
7	1	+ 1,2	+ 0,32	0,64	–0,2	+0,52	0,62
7	2	– 0,7	+ 1,09	0,64	–0,04	+0,52	0,62
7	3	+ 1,2	+ 0,70	0,64	–0,8	–0,19	0,64
7	4	– 0,6	– 1,27	0,64	+0,2	+0,09	0,64
7	5	– 0,6	+ 0,07	0,61	–0,1	+0,11	0,64
7	6	– 1,8	– 0,30	0,60	+0,1	+0,10	0,61
7	7	+ 0,4	+ 0,36	0,60	–0,9	+0,05	0,61
8	0	+ 0,6	+ 0,96	0,52	–	–	–
8	1	+ 1,0	+ 0,08	0,54	+0,1	+0,39	0,54
8	2	+ 0,6	+ 0,60	0,53	+1,0	+0,60	0,53
8	3	– 0,8	– 0,06	0,53	–0,4	+0,22	0,54
8	4	– 0,5	– 0,51	0,54	+0,8	+0,10	0,53
8	5	– 0,8	– 0,65	0,53	+0,3	+0,56	0,54
8	6	– 1,1	+ 0,03	0,51	+1,1	+0,90	0,53
8	7	+ 0,3	+ 1,0	0,51	+0,2	+0,45	0,51
8	8	+ 0,7	– 0,78	0,51	–0,1	–0,17	0,51

*Tr. Note: Commas indicate decimal points.

As may be seen from the figures and tables given in this appendix, the coefficients for the 5th and 6th zonal harmonics are extremely uncertain, while the coefficients for the sectoral and tesseral harmonics have been determined with still less certainty. Moreover, the same may be said of the coefficients for the upper harmonics. For this reason, there is little justification in using a gravitational potential which includes harmonics above the fourth at the present time. /41

As has already been mentioned previously, knowing the external gravitational field is equivalent to knowing the figure of the Earth and vice versa. Actually, the Stokes formula or Stokes series may be used for going from the equation which describes the gravitational field to that which describes the shape of the Earth [36, 37]. Maps for the altitudes of the geoid given by various authors (Zhongolovich 1952, Kaula 1961) differ considerably from one another [37]. This is explained , just as the difference between the coefficients in the expansion calculated by the different authors, by the differences in the initial material and the methods for analyzing this material.

The representation of the discrepancy between the geoid and the terrestrial spheroid (the figure represented by the sum of harmonics P_{00} and P_{20}) gives a table which was presented by Kaula [43] for the standard deviations $\sigma_n\{N\}$ of altitudes corresponding to the n-th harmonic of the equation for the surface.

TABLE 3 *

n	σ_n^2, м^2	n	σ_n^2, м^2	n	σ_n^2, м^2	n	σ_n^2, м^2
2	308	10	7,8	18	2,8	26	0,7
3	406	11	7,6	19	1,3	27	0,3
4	140	12	2,4	20	0,8	28	0,5
5	28	13	4,4	21	1,5	29	0,2
6	41	14	5,8	22	1,0	30	0,1
7	3,3	15	4,8	23	0,8	31	0,05
8	19,6	16	1,1	24	0,9	32	0,09
9	14,6	17	2,0	25	0,7		

The overall variance for the difference in altitudes of the geoid and spheroid is about

$$1{,}075 \text{ m}^2, \text{ and } \sum_{n=2}^{32} \sigma_n^2 = 1{,}062 \text{ m}^2 .$$

*Tr. Note: Commas indicate decimal points.

Thus, the mean difference between the altitudes of the geoid and of the figure represented by 32 harmonics is equal to approximately 3.6 meters. It may be seen from Table 3 that $\sum_{n=2}^{4} \sigma_n^2 = 908\ m^2$, i.e. the remaining harmonics (from the 5th to the 32nd) have about the same overall effect on the shape of the geoid as the fourth alone. /42

This conclusion is completely applicable to the relative weight of the various harmonics in the expansion of the Earth's gravitational potential. Everything said here leads us to assume that the upper harmonics make a small contribution to the disturbing function, and a disregard for these harmonics (due specifically to inaccurate knowledge of the coefficients) should not cause any serious discrepancies between the calculated and actual motion of an artificial satellite. However, in many cases, it is necessary to have at least an approximate numerical estimate of this value. The derivation of such an estimate is taken up in the fourth section.

§4. Approximate Description of the Gravitational Field

In all cases where the gravitational potential is represented in the form of a finite sum (and this must necessarily be done in practice), it is necessary to evaluate the error due to dropping the remainder of the series[1].

Before going on to an estimate of the error introduced by averaging of this type, let us mention the important property of orthogonality of spherical functions, according to which

$$\left.\begin{aligned} &\int_{\Sigma} P_{nm}(\sin\varphi) P_{ij}(\sin\varphi) \begin{matrix}\cos\\ \sin\end{matrix} m\lambda \begin{matrix}\cos\\ \sin\end{matrix} j\lambda\, ds = 0 \quad \text{when} \quad n \neq i,\ m \neq j; \\ &\int_{\Sigma} \left[P_{nm}(\sin\varphi) \begin{matrix}\cos\\ \sin\end{matrix} m\lambda\right]^2 ds = \frac{2\pi\delta_m}{2n+1} \frac{(n+m)!}{(n-m)!} \quad \text{when} \quad n = i,\ m = j. \end{aligned}\right\} \qquad (4.1)$$

Here $\delta_m = 2$ when $m = 0$, and $\delta_m = 1$ when $m > 0$. Integration is carried out with respect to the surface of the sphere Σ.

Since the upper harmonics in the expansion for the potential V describe small (low-amplitude) local singularities in the gravitational field, disregarding them is equivalent to averaging the function V. This smooths out the oscillating nature of V since the localized structure of the field is ignored while its fundamental governing principles are maintained.

[1] An estimate of the residual term of the expansion $1/\Delta$ (see (III.2')) may be found, for instance [31, 44, 45]. However, obtaining this estimate does not solve the problems raised in this section.

The quantity $\frac{2\pi\delta_m}{2n+1} \cdot \frac{(n+m)!}{(n-m)!}$ is called the norm of the function $P_{nm}(\sin\varphi)\,{}^{\cos}_{\sin}\, m\lambda$. A proof of property (4.1) may be found [31, 44, 45].

Let us now consider a potential of the form /43

$$V_B = \frac{\mu}{r}[1 + q^2 c_{20} P_{20}(\sin\varphi)], \qquad (4.2)$$

where $q = r_0/r < 1$.

The function $V_{20} = \frac{\mu}{r} q^2 c_{20} P_{20}(\sin\varphi)$ in expression (4.2) is the disturbing potential. The mean square value of V_{20} on sphere Σ (with regard to (4.1) is equal to

$$\sigma_{20} = \sqrt{\frac{\int_\Sigma V_{20}^2 ds}{4\pi r^2}} = \sqrt{\frac{1}{5}}\,\frac{\mu}{r}\, q^2\, |c_{20}|. \qquad (4.3)$$

Reducing this value to the potential of the spherical Earth, we get

$$\bar{\sigma}_{20}^{*} = \sqrt{\frac{1}{5}}\, q^2\, |c_{20}|. \qquad (4.3')$$

The nature of relationship $\bar{\sigma}_{20}(q)$ is shown in Fig. 1, which indicates the part of the potential of the spherical Earth which is comprised by disturbing potential V_{20} as a function of the relative distance $1/q$.

Let us now break up the gravitational potential of the Earth into two parts. The first, which will be taken into account, we shall call the /44 normal potential; the second, which will be disregarded in analysis, we shall call the disturbing potential (or the potential of gravitational anomalies). The potential of anomalies thus represents the difference between the potentials of the true and normal fields[1].

[1] The normal potential has already been considered above in §3. It was selected so that the disturbing potential was small enough to permit solving the gravimetric problem in the linear formulation. In the given case, as in general in the theory of satellite motion, the normal potential is selected so that the errors in determining the motion of the center of gravity of the satellite will fall within certain limits.

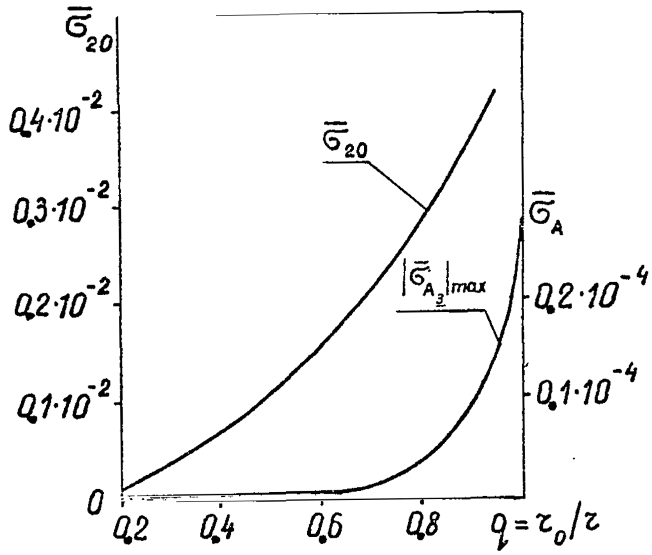


Fig. 1.

The gravitational field of the general terrestrial ellipsoid is taken as normal field in the theory of the Earth's figure. In problems associated with satellite motion, a different normal gravitational field may be chosen. In this case, two goals are kept in mind: maximum simplicity, and achievement of the required accuracy in solving the problem.

Let the normal potential be represented by the sum of the first (k - 1) spherical harmonics. Then the potential of anomalies is written in the form

$$T = \frac{\mu}{r} \sum_{n=k}^{\infty} \sum_{m=0}^{n} \left(\frac{r_0}{r}\right)^n (c_{nm} \cos m\lambda + d_{nm} \sin m\lambda) P_{nm}(\sin\varphi) \tag{4.4}$$

or

$$V_A = \frac{\mu}{r^2} \sum_{n=k}^{\infty} q^{n-1} N_n .$$

Here

$$N_n = \sum_{m=0}^{n} r_0 (c_{nm} \cos m\lambda + d_{nm} \sin m\lambda) P_{nm}(\sin\varphi)$$

are the deviations in the altitude of the geoid corresponding to the Legendre polynomial P_n.

Hence an expression may be derived for the mean square value of V_A:

$$\sigma_A^2 = \int_\Sigma V_A^2 ds \Big/ 4\pi = \frac{1}{4\pi} \frac{\mu^2}{r^4} \sum_{n=k}^{\infty} q^{2(n-1)} \sigma_n^2 .$$

According to Schwartz's inequality, we get /45

$$\sigma_A^2 \leq \frac{\mu^2}{4\pi r^4}\sqrt{\sum_{n=k}^{\infty} q^{4(n-1)} \sum_{n=k}^{\infty} \sigma_n^4}\,.$$

Since

$$\sum_{n=k}^{\infty} q^{4(n-1)} = \frac{1}{q^4}\left[\sum_{n=1}^{\infty} q^{4n} - \sum_{n=1}^{k-1} q^{4n}\right] = \frac{q^{4(k-1)}}{1-q^4}\,,$$

we have, for the estimate reduced to the potential of the spherical Earth

$$\bar{\sigma}_A \leq \frac{1}{2r^2\sqrt{\pi}} \frac{q^{k-1}}{\sqrt[4]{1-q^4}} \sqrt[4]{\sum_{n=k}^{\infty} \sigma_n^4}\,. \tag{4.5}$$

Since the right member of inequality (4.5) contains an infinite sum of quantities σ_n^4, this expression is also unsuitable for practical use. However, this estimate may find application if the infinite series is replaced by a partial sum with a sufficiently large number of terms.

In the case where the number of terms in this sum is equal to i, a repetition of the above operations gives

$$\sigma_A^2 \leq \frac{\mu^2}{4\pi r^4}\sqrt{\sum_{n=k}^{i} q^{4(n-1)} \sum_{n=k}^{i} \sigma_n^4} = \frac{\mu^2}{4\pi r^4}\sqrt{\frac{q^4(q^{k-1}-q^i)}{1-q^4} \sum_{n=k}^{i} \sigma_n^4}\,.$$

The relative error $\bar{\sigma}_A$ is equal to

$$\bar{\sigma}_A \leq \frac{q}{2r^2\sqrt{\pi}} \sqrt[4]{\frac{q^{k-1}-q^i}{1-q^4} \sum_{n=k}^{i} \sigma_n^4}\,. \tag{4.5'}$$

It was pointed out in §3 that the mean difference between the altitudes of the geoid (the figure represented by an infinite series) and that of the figure described by 32 harmonics is approximately 3.6 m (according to

Kaula). Therefore, i may be taken as equal to 32 in the formulas given above with an accuracy completely suitable for practical purposes. In this case the values of σ_n^4 are taken from Table 3, and the lower limit of the summations is equal to n = k = 3.

The deviations σ_n in Table 3 are determined relative to an ellipsoid described by only the zeroth and second zonal harmonics; therefore, the presence of the second sectoral harmonic must be taken into account in the right-hand member of inequality (4.5). The deviation corresponding to this harmonic /46
is given in the same table by the quantity σ_2 (the notation $\sigma_{22} = \sigma_2$ is used in inequality (4.6)).

In the case where the potential of anomalies contains higher harmonics beginning with the second sectoral harmonic estimate (4.5) takes the form

$$\bar{\sigma}_{A_3} \leq \frac{q}{2r^2\sqrt{\pi}} \sqrt{\sigma_{22}^2 + \sqrt{\frac{q^2 - q^{32}}{1-q^4} \sum_{n=3}^{32} \sigma_n^4}}, \qquad (4.6)$$

while if it contains harmonics beginning with the k-th,

$$\bar{\sigma}_{A_k} \leq \frac{q}{2r^2\sqrt{\pi}} \sqrt[4]{\frac{q^{k-1} - q^{32}}{1 - q^4} \sum_{n=k}^{32} \sigma_n^4}. \qquad (4.6')$$

When Table 3 is used for giving the quantities σ_n^4, the right-hand members of inequalities (4.6), (4.6') should be increased by a factor of 2-2.5.

The curve $|\bar{\sigma}_{A_3}|_{max}$ in Fig. 1 represents the maximum relative mean square error in the case where the normal potential contains only the zeroth and second zonal harmonics.

Table 4 shows how this error changes as a function of the number of harmonics retained in the normal potential.

TABLE 4 *

Higher Harmonic in the Normal Potential	$\lvert\sigma_{A_k}\rvert_{max}$	
	$q = 0.9$, $r_{av} = 7080$ km	$q = 0.68$, $r_{av} = 9360$ km
2[1]	$0{,}1\cdot10^{-4}$	$0{,}14\cdot10^{-6}$
4	$0{,}1\cdot10^{-5}$	$0{,}3\cdot10^{-7}$
8	$0{,}8\cdot10^{-7}$	$0{,}16\cdot10^{-7}$
16	$0{,}2\cdot10^{-7}$	$0{,}3\cdot10^{-8}$

Perturbations in motion may be assumed to be linearly dependent on the magnitude of the disturbing potential with sufficient accuracy for practical purposes. Since disturbances from the effect of the second zonal harmonic are comparatively easily found, $\bar{\sigma}_{A_k}$ may be compared with $\bar{\sigma}_{20}$ at various k to estimate (in fractions of the disturbances from the second harmonic) the errors in motion which result from disregarding gravitational anomalies. /47

It will be evident from the results of the second chapter that disturbances from the second zonal harmonic (over an interval of up to a few days) may reach several hundred kilometers. The error due to disregarding terms of the potential expansion associated with $\bar{\sigma}_{A_3}$ is two orders of magnitude less than the quantity $\bar{\sigma}_{20}$, therefore the error in the position of a nearby satellite ($q \simeq 0.9$) will be no more than a few kilometers in this case. When four spherical harmonics are considered in the potential expansion (for the same values of q,), the error will be three orders of magnitude lower, i.e. no more than a few hundred meters off in the position of the satellite. The inclusion of 8 or 16 spherical harmonics in the potential would make it possible to determine the position of a satellite close to the Earth with extremely high precision if the coefficients associated with the higher harmonics were known with sufficient certainty.

An uncertainty in the position of these satellites is also introduced by the fact that the effect of other insufficiently studied factors (such as variations in the density of the upper atmosphere) may lead to errors of the same order of magnitude.

These conditions dictate limitation of further analysis to a potential containing no more than four spherical harmonics. The order of the resultant error in this case may be estimated by using the inequalities already derived.

---

[1] Harmonic P_{22} is included in the potential of anomalies.

* Tr. note: Commas indicate decimal points.

We must not lose sight of the fact that the effect of higher terms in the potential expansion (chiefly the terms which account for the triaxial form of the Earth) may cause resonance phenomena in the motion of a satellite. This point is taken up in more detail in Appendix VIII to the second chapter.

§5. Models of the Gravitational Field

Let us write out the equation for the family of equipotential surfaces of a rotating attractive body

$$W = V + \frac{1}{2}\omega^2 r^2 \cos^2\varphi = C, \tag{5.1}$$

here V is series (2.6').

Some value of the constant C isolates from the set of equipotential surfaces that one which is the surface of the attractive body in the case where that body is a liquid.

Let us determine the coefficients of the expansion V and the constant C assuming that the Earth is an ellipsoid of revolution /48

$$r^2 \left[\frac{\cos^2\varphi}{r_0^2} + \frac{\sin^2\varphi}{r_0^2(1-\alpha)^2}\right] = 1, \tag{5.2}$$

with only a slight deviation from the circumscribed sphere of radius r_0. The deviation of surface (5.2) from the spherical is due to the presence of a small parameter α, which as a rule is taken to be polar flattening

$$\alpha = \frac{a - b}{a},$$

where a and b are the semimajor and semiminor axes of the ellipse; the semiminor axis coincides with the axis of rotation of the Earth.

The eccentricity of the ellipse (the so-called first eccentricity)

$$e = \sqrt{a^2 - b^2}/a$$

is related to flattening α by the expression[1]

$$e = \sqrt{\frac{a+b}{a-b}\alpha}$$

or [32]

$$e^2 = 2\alpha + 3\alpha^2 + 4\alpha^3 + \dots .$$

From equation (5.2) we get

$$r^2 = \frac{r_0^2(1-\alpha)^2}{(1-\alpha)^2\cos^2\varphi + \sin^2\varphi}. \tag{5.3}$$

By transforming the denominator

$$(1-\alpha)^2\cos^2\varphi + \sin^2\varphi = 1 - 2\alpha\cos^2\varphi + \alpha^2\cos^2\varphi + \\ + \alpha^2\cos^4\varphi - \alpha^2\cos^4\varphi = (1-\alpha\cos^2\varphi)^2 + \alpha^2\cos^2\varphi\sin^2\varphi,$$

we get

$$r^2 = \frac{r_0^2(1-\alpha)^2}{(1-\alpha\cos^2\varphi)^2 + \alpha^2\cos^2\varphi\sin^2\varphi}.$$

Alternatively /49

$$r = \frac{r_0(1-\alpha)}{1-\alpha\cos^2\varphi} = \left[1 + \frac{\alpha^2\cos^2\varphi\sin^2\varphi}{(1-\alpha\cos^2\varphi)^2}\right]^{-1/2}.$$

---

[1] In addition to the first eccentricity e_1, we might also consider the second eccentricity $e_2 = \sqrt{a^2-b^2}/b$. The relationship between e_1 and e_2 is: $e_1^2 = e_2^2/(1+e_2^2)$.

Let the flattening of the terrestrial ellipsoid be so small that we may disregard powers of α greater than the second. Then, if we expand the expression enclosed in the brackets in a series, retaining only terms of the first and second negative orders of magnitude, we get

$$r = \frac{r_0(1-\alpha)}{1-\alpha\cos^2\varphi}\left[1-\frac{1}{2}\alpha^2\cos^2\varphi\sin^2\varphi\right].$$

Hence,

$$r_0/r = 1+\alpha\sin^2\varphi+\frac{3}{2}\alpha^2\sin^2\varphi-\frac{1}{2}\alpha^2\sin^4\varphi,$$

which gives within the assumed limits of accuracy

$$r_0/r = 1+\alpha\sin^2\varphi+\frac{3}{2}\alpha^2\sin^2\varphi-\frac{1}{2}\alpha^2\sin^4\varphi. \tag{5.4}$$

The gravitational potential of a uniform ellipsoid of revolution may be expressed in terms of zonal harmonics alone (see §2) in the form

$$V = \frac{\mu}{r}+\frac{\mu}{r}\sum_{n=1}^{\infty}\left(\frac{r_0}{r}\right)^{2n} c_{2n_0}P_{2n_0}(\sin\varphi). \tag{5.5}$$

The deviation of ellipsoid (5.3) from the spherical is due to the presence of parameter α, while the deviation of potential (5.5) from that for a sphere is due to the presence of the sum in the right-hand member. Therefore, the coefficients assoicated with the terms of this sum (beginning with c_{20}) will be quantities of at least order $O(\alpha)$[1].

Further, we see that this order pertains only to coefficient c_{20}; coefficient c_{40} has order $O(\alpha^2)$, c_{60} has order $O(\alpha^3)$, etc. (see [32]).

---

[1] By convention, the symbol $O(\alpha)$ denotes a small quantity of order α. The symbol $o(\alpha)$ designates a quantity of a higher negative order of magnitude than α.

Thus, we may write (5.5) in the form

$$V=\frac{\mu}{r_0}\left[\frac{r_0}{r}+\left(\frac{r_0}{r}\right)^3 c_{20}P_{20}(\sin\varphi)+\left(\frac{r_0}{r}\right)^5 c_{40}P_{40}(\sin\varphi)\right], \tag{5.6}$$

retaining only the two first terms from the sum in (5.5) and assuming for the present that they are both of order $0(\alpha)$.

Let us introduce the notation

$$\omega^2 r_0^3(1-\alpha)/\mu = m.$$

The quantity m is approximately equal to the ratio (m') of centrifugal force to acceleration due to gravity at the equator. The parameter /50

$$m' = \omega^2 r_0^3/\mu = \omega^2 r_0/g_{EQ}$$

which is of the order of flattening, is called the parameter of the Earth's figure. Actually, if it is assumed that $\omega = 2\pi/86{,}164.1$ (mean seconds in sidereal days), $r_0 = 6{,}378{,}245$ meters, $g_{EQ} = 978.049$ gals, we get

$$m = 1{:}288.365.$$

This value coincides almost exactly with the latest determinations of polar flattening of the Earth.

The potential of centrifugal force with regard to terms down to $0(\alpha^2)$ may be transformed as follows:

$$\begin{aligned}\frac{1}{2}\omega^2 r^2\cos^2\varphi &= \frac{1}{2}\frac{\mu m}{r_0^3(1-\alpha)}r_0^2(1-\alpha)^2(1+2\alpha\cos^2\varphi)\cos\varphi = \\ &= \frac{1}{2}\frac{\mu m}{r_0}[1+\alpha-(1+3\alpha)\sin^2\varphi+2\alpha\sin^4\varphi].\end{aligned} \tag{5.7}$$

Taking expressions (5.4), (5.6) and (5.7) into account, expression (5.1) becomes

$$\frac{\mu}{r_0}\left\{1+\left(\alpha+\frac{3}{2}\alpha^2\right)\sin^2\varphi-\frac{1}{2}\alpha^2\sin^4\varphi+(1+\right.$$

$$+3\alpha\sin^2\varphi)c_{20}P_{20}(\sin\varphi)+(1+5\alpha\sin^2\varphi c_{40}P_{40}(\sin\varphi)+$$

$$\left.+\frac{1}{2}m[1+\alpha-(1+3\alpha)\sin^2\varphi+2\alpha\sin^4\varphi]\right\}+o(\alpha^2)=C.$$

Expanding the expressions for Legendre's polynomials, we get within the assumed limits of accuracy

$$\frac{\mu}{r_0}\left\{1+\left(\alpha+\frac{3}{2}\alpha^2\right)\sin^2\varphi-\frac{1}{2}\alpha^2\sin^4\varphi+(1+3\alpha\sin^2\varphi)(1+\right.$$

$$+3\alpha\sin^2\varphi)\frac{1}{2}c_{20}+(1+5\alpha\sin^2\varphi)(35\sin^4\varphi-30\sin^2\varphi+3)\frac{1}{8}c_{40}+$$

$$\left.+\frac{1}{2}m[1+\alpha-(1+3\alpha)\sin^2\varphi+2\alpha\sin^4\varphi]\right\}=C.$$

Since the expression on the left-hand side of the equation must remain constant, the coefficients associated with the various powers of sin φ must be set equal to zero after removing parentheses and combining similar terms.

As a result, we get the four equations

$$\left.\begin{array}{l}
C=\frac{\mu}{r_0}\left[1-\frac{1}{2}c_{20}+\frac{3}{8}c_{40}+\frac{1}{2}m\right];\\
\frac{3}{2}c_{20}-\frac{30}{8}c_{40}+\alpha+\frac{3}{2}\alpha^2-\frac{3}{2}\alpha c_{20}-\frac{1}{2}m-\frac{3}{2}m\alpha+\frac{15}{8}\alpha c_{40}=0;\\
\frac{35}{8}c_{40}-\frac{1}{2}\alpha^2+\frac{9}{2}\alpha c_{20}-\alpha m-\frac{150}{8}\alpha c_{40}=0;\\
\frac{175}{8}\alpha c_{40}=0.
\end{array}\right\} \qquad (5.8)$$

The last equation shows that $\alpha c_{40} = 0$ within the assumed limits of accuracy, i.e. the quantity $c_{40} = o(\alpha)$.

It is evident from the first equation in system (5.8) that the constant C in the equation for the surface of the terrestrial ellipsoid (5.1) is equal to the potential of the force of gravity at the equator.

From the second equation in system (5.8), limiting ourselves to the terms of order $O(\alpha)$, we get

$$c_{20}=-\frac{2}{3}\left(\alpha-\frac{1}{2}m\right). \qquad (5.9)$$

From the third, we get

$$c_{40} = \frac{8}{35}\left(\frac{7}{2}\alpha^2 - \frac{5}{2}\alpha m\right). \qquad (5.10)$$

Substituting (5.10) in the second equation of system (5.8), we get, taking terms of order $O(\alpha^2)$ into account

$$c_{20} = -\frac{2}{3}\left(\alpha - \frac{1}{2}m - \frac{1}{2}\alpha^2 + \frac{1}{7}\alpha m\right). \qquad (5.11)$$

Thus, the first two coefficients of the expansion for the disturbing potential are expressed in terms of the parameters for the terrestrial ellipsoid. It is clear from these expressions that $c_{20} = O(\alpha)$, while $c_{40} = O(\alpha^2)$. It has already been pointed out above that $c_{60} = O(\alpha^3)$ [32]. The remaining coefficients associated with the zonal harmonics are of a higher negative order of magnitude.

Let us now examine various models of the gravitational field of the Earth. By expressing the potential for practical purposes in the form of a finite sum rather than by series (2.6') we allow an error which decreases with an increase in the number of terms retained (see the preceding section). However, since any definite effect of the gravitational field may be described by a combination of several harmonics, the addition of one or several terms does not always reduce the error with which the true field is approximated.

For instance, the most significant term (in the expression for the potential) which reflects the phenomenon of triaxiality (equatorial flattening) is /52

$$\frac{\mu}{r}\left(\frac{r_0}{r}\right)^2 (c_{22}\cos 2\lambda + d_{22}\sin 2\lambda) P_{22}(\sin\varphi). \qquad (5.9')$$

If we add to the potential represented by the sum of the zeroth and second zonal harmonics only one of the terms of expression (5.9') (which will by no means account for equatorial flattening), there will be absolutely no improvement in the description of the gravitational field of the Earth.

Therefore, the geoid (and consequently its gravitational field) should be approximated by actual physical bodies, with consideration given to the negative order of magnitude of the terms which are dropped, and of those which are retained.

In the discussion which follows, we shall call such approximate representations of the Earth and its potential "models".

Several simplest models may be constructed.

The Spherical Earth with potential

$$V_0 = \frac{\mu}{r}. \tag{5.10'}$$

This model is convenient in the fact that motion in field (5.10') is most simply described. As we know [1, 2, 3], an orbit in this case is determined by 5 constants (Kepler elements) which are expressed in terms of the initial conditions and do not change throughout the entire extent of the motion. Motion in a field of potential V_0 is called Keplerian motion. The difference between the potential of the geoid and the potential of a sphere

$$\Delta V = V - V_0$$

is called the disturbing potential in analyzing satellite motion, while the corresponding motion, which differs little from Keplerian motion, is called disturbed motion.

In order to determine the position of a satellite in orbit in the case of Keplerian motion, it is sufficient to perform a single quadrature or to solve a transcendental equation (Kepler's equation).

The errors in satellite motion which arise when model (5.10') is used will be discussed in the second chapter. In a number of instances (for example in preliminary calculations associated with orbit planning [4]), these errors may be disregarded for the sake of simplicity and clarity of the computations.

The Spherical Earth with potential (model Б) /53

$$V_{Б} = \frac{\mu}{r}\left[1 + c_{20}\left(\frac{r_0}{r}\right)^2 P_{20}(\sin\varphi)\right]. \tag{5.11'}$$

According to (5.9), coefficient c_{20} is proportional to polar flattening of the Earth. Therefore, (5.11') is the potential of an ellipsoid of revolution in which the square of the flattening may be disregarded. We shall call such an ellipsoid, which differs little from a sphere, a "spheroid"[1]. If the expression for P_{20} $(\sin\varphi)$ is expanded in (5.11'), we see that the second term in the brackets will be equal to zero when $\sin^2\varphi = 1/3$, which corresponds to two latitudes $\varphi = \pm 35°15'52''$. At these points, the potential of attraction of a spheroid is equal to the potential of attraction of a sphere. At the equator ($\varphi = 0$) and at the poles ($\varphi = 90°$), the potential of the spheroid is respectively greater than and less than that of the sphere. The maximum difference between spheroid and sphere is proportional to flattening ($\alpha = 1:298.3$; see §3 and also [37]), i.e. it is equal to approximately 21 km.

It was shown above that the second zonal harmonic is the most significant part of the disturbing potential. Therefore, (see second chapter), the errors in satellite motion when using model $V_Б$ will be considerably lower than in the case of model V_0.

In cases where satellite motion is considered over an interval of several orbits, the selection of attraction potential in form (5.11') quite frequently yields the required accuracy of the solution.

The Earth is represented by an ellipsoid of revolution with potential (model B)

$$V_B = \frac{\mu}{r}\left[1 + c_{20}\left(\frac{r_0}{r}\right)^2 P_{20}(\sin\varphi) + c_{40}\left(\frac{r_0}{r}\right)^4 P_{40}(\sin\varphi)\right]. \qquad (5.12)$$

Ellipsoid of revolution in this case is understood to mean a body for which the degrees of flattening $o(\alpha^2)$ may be disregarded. /54

[1] This is the name used in the theory of the Earth's figure for a body having potential $V_Б$. In the literature on celestial mechanics, the term "spheroid" is sometimes understood to mean an ellipsoid of revolution for which the gravitational potential is the sum of a finite or infinite number of zonal spherical harmonics. A spheroid for which the potential is determined in the form of equation (5.11') while c_{20} is determined from equation (5.9), coincides to an accuracy of first-order infinitestimals with the Clairaut spheroid which is part of the theory of the Earth's figure.

Since terms of the expansion are taken into account up to degrees α^2 in (5.12), coefficients c_{40} and c_{20} must be taken in form (5.10), (5.11)[1].

The square of flattening should be taken into consideration from the standpoint of improving accuracy in solving a satellite motion problem, as will be evident from the results of the second chapter, chiefly when considering comparatively long time intervals.

The Earth is represented as a triaxial ellipsoid (model Γ)

$$V_\Gamma = \frac{\mu}{r}\left[1 + c_{20}\left(\frac{r_0}{r}\right)^2 P_{20}(\sin\varphi) + \left(\frac{r_0}{r}\right)^2 (c_{22}\cos 2\lambda + d_{22}\sin 2\lambda) P_{22}(\sin\varphi)\right]. \tag{5.13}$$

The effect of equatorial flattening γ is characterized in equation (5.13) by the second sectoral harmonic, coefficients c_{22} and d_{22} being proportional to γ.

Since

$$\gamma = 1:30{,}000,$$

c_{22} and d_{22} are approximately of the same order of magnitude as the square of polar flattening α.

The difference between the semimajor and semiminor axes of the equatorial ellipse is approximately 150-250 meters.

According to [37], the semimajor axis of this ellipse has an extremely uncertain longitude which lies between $\lambda_1 = 38°$ and $\lambda_2 = -25°$.

Let us note that I. D. Zhongolovich [35] gives $\gamma = 1:32{,}000$ and the meridians of least flattening as $\lambda_1 = +84°$ and $\lambda_2 = -96°$.

The Earth is represented as a symmetric spheroid with potential

[1] Actually, in both the preceding case and in this one, the same numerical value of c_{20} is taken, which is determined as a rule with consideration to powers of α higher than the first.

(Model Д)

$$V_{Д} = \frac{\mu}{r}\left[1 + c_{20}\left(\frac{r_0}{r}\right)^2 P_{20}(\sin\varphi) + c_{30}\left(\frac{r_0}{r}\right)^3 P_{30}(\sin\varphi)\right]. \quad (5.14)$$

According to §2, effects of asymmetry should be represented by the odd zonal harmonics, the third being the highest in amplitude. The coefficient associated with this harmonic also has the order of the square of polar flattening. According to available data, the northern polar radius of the Earth is somewhat longer than the southern polar radius. This difference is equal to 70 meters [35]. /55

When the data of [32] are used, the asymmetry factor has extremely little effect on the motion of Earth Satellites (when coefficients determined on the basis of more complete and more recent information as to the gravitational field are used, the asymmetry factor shows up more strongly, and even exceeds the effect of triaxiality).

The Earth is represented as a triaxial asymmetric ellipsoid with potential (model E)

$$V_E = \frac{\mu}{r}\left\{1 + \left(\frac{r_0}{r}\right)^2 [c_{20} P_{20}(\sin\varphi) + (c_{22}\cos 2\lambda + d_{22}\sin 2\lambda) P_{22}(\sin\varphi)] + \left(\frac{r_0}{r}\right)^3 c_{30} P_{30}(\sin\varphi) + \left(\frac{r_0}{r}\right)^4 c_{40} P_{40}(\sin\varphi)\right\}. \quad (5.15)$$

This is the most complete model of all those considered. It gives a better approximation of the geoid than any of the others.

Actually, expression (5.15) contains all principal terms of the expansion for the gravitational potential written for $n \leq 4$. The omitted tesseral and sectoral harmonics of third and fourth degrees are small since the coefficients associated with them are of order $o(\alpha^2)$.

Each of the models constructed here may be selected as a model of the Earth's potential in solving problems associated with satellite motion. In this case, the selected model may be called the normal gravitational potential, while the difference between this model and the potential of the geoid may be called the anomalous gravitational potential or gravitational anomalies. Selection of the normal potential is determined by the problem to be formulated. The accuracy of the given models is evaluated in the next chapter.

"But are we right in assuming the hypothesis of central forces? Is this hypothesis strictly accurate?"

H. Poincaré

"Not one of the theories is true to a greater extent than the map is a true image of the country. And how convenient to have different maps (various scales) to study the geography of the country, how convenient to have several diagrams and models of nature."

J. L. Synge

Chapter Two

DISTURBED MOTION OF AN ARTIFICIAL EARTH SATELLITE

In this chapter we shall consider the disturbed motion of an artificial Earth satellite over a time interval of no more than a few days. An analytical investigation of the orbits of circular and nearly circular satellites is given in Appendices VI, VII and VIII, as well as an investigation of disturbances in motion over a very long interval (about one hundred satellite orbits).

The relegation of this material to the appendices is explained by the fact that the equations used in the analysis are derived later, in the third chapter.

§6. Description of Disturbed Motion

Disturbed motion of artificial satellites is described by differential equations of form (0.2). The most convenient parameters for numerical and qualitative analysis of disturbances are however, not the rectangular or any /57
other coordinates, but rather the osculating elements since each of them directly characterizes either the geometry or the kinematics of the motion.

Several equivalent systems of osculating elements conventionally used in celestial mechanics are known [1, 2, 3]. Each of them involves orbital eccentricity e and angular distance of the perigee ω or some other parameter which determines the position of the perigee of the orbit.

In the case of small eccentricities, a small divisor 1/e appears in the osculating elements on the right side of the differential equations, while for eccentricities equal to zero, the quantity ω generally becomes indeterminate. This makes it difficult to describe the osculating motion of satellites with low orbital eccentricity.

However, new systems of osculating parameters which differ from those assumed in classical celestial mechanics[1] and which are free from the given disadvantage may be constructed on the basis of the first integrals of the equations of motion.

We shall use here one of the possible systems of this type in which the eccentricity and angular distance of the perigee are replaced by the two Laplace vector components [1 - 3]q and k lying in the plane of the osculating orbit[2] [46]. In this case, they are simply expressed in terms of e and ω:

$$\left.\begin{aligned} q &= e\cos\omega; \\ k &= e\sin\omega, \end{aligned}\right\} \tag{6.1}$$

and also readily permit the reverse transformation:

$$e = \sqrt{q^2 + k^2}; \quad \omega = \arcsin k/e \ \text{when} \ \omega = \arccos q/e. \tag{6.2}$$

If functions q and k are found by using numerical integration of differential equations, it is better (in view of unavoidable computational inaccuracy) to determine the quantity ω as the arithmetical mean

$$\bar{\omega} = \frac{1}{2}\left(\arcsin\frac{k}{e} + \arccos\frac{q}{e}\right). \tag{6.2'}$$

If the functions q and k are used in the system of osculating elements, /58
there is no need for a transition to variables e and ω in solving practical problems. However, such a transition may be advisable in studying orbits since the latter two parameters are geometrically more graphic.

In the case of orbits with initial eccentricity equal to zero ($e_0 = 0$), the disturbances $\delta\omega$ may be found from the following equations:

---

[1] Classical celestial mechanics in this case is understood to mean the celestial mechanics of natural heavenly bodies which has been developed over the course of several centuries.

[2] The quantities q and k are actually the components of the Laplace vector /57
divided by the constant μ.

$$\delta\omega = \omega = \frac{\pi}{2}N + \bar{\omega}M;$$

$$N = (1-\operatorname{sgn} q) + (1-\operatorname{sgn} q \operatorname{sgn} k)(1+\operatorname{sgn} q);$$

$$M = \operatorname{sgn} q \operatorname{sgn} k.$$

The disturbances δe in this case are determined from the first relationship (6.2) (since $\delta e = e$ when $e_0 = 0$).

When $e_0 \neq 0$, disturbances δe and $\delta\omega$ are calculated in the form

$$\delta\omega = \frac{1}{e}(\cos\omega\,\delta k - \sin\omega\,\delta q);$$

$$\delta e = \frac{1}{e}(k\delta k + q\delta q).$$

The quantities e, ω, q and k are taken in the instant (period) preceding osculation, i.e. the values taken for these quantities are those which are used as the reference values in computing disturbances δq and δk.

It will be possible to calculate the motion of a satellite with the aid of variables q and k if an expression is found for them in terms of the kinematic parameters of the orbit.

In order to derive these relationships, we write the identity for the radial and transversal components of velocity:

$$V_r = \frac{dr}{dt} = \frac{dr}{du}\cdot\frac{du}{dt}; \quad V_\tau = r\frac{du}{dt}.$$

Using the integral of areas [1, 2, 3] $r^2 du/dt = r V_\tau = \sqrt{\mu p} = C$, we get

$$V_r = \frac{\sqrt{\mu p}}{r^2}\frac{dr}{du}; \quad V_\tau = \frac{\sqrt{\mu p}}{r}.$$

Since

$$\frac{dr}{du} = \frac{r^2}{p}(q\sin u - k\cos u),$$

then

$$\left.\begin{aligned} \sqrt{\frac{p}{\mu}}\, V_r &= q \sin u - k \cos u; \\ \sqrt{\frac{p}{\mu}}\, (V_\tau - 1) &= q \cos u + k \sin u. \end{aligned}\right\}$$ /59

Here p is the focal parameter of the orbit; u is the increment in latitude of the satellite.

From this system of equations we get the desired relationships:

$$\left.\begin{aligned} q &= \sqrt{\frac{p}{\mu}}\, V_r \sin u + \left(\sqrt{\frac{p}{\mu}}\, V_n - 1\right) \cos u; \\ k &= \left(\sqrt{\frac{p}{\mu}}\, V_\tau - 1\right) \sin u - \sqrt{\frac{p}{\mu}}\, V_r \cos u. \end{aligned}\right\} \tag{6.3}$$

Thus, if we know the initial values of the components of velocity V_r and V_τ, as well as p and u, we may use (6.3) to find the initial values of q_0 and k_0, which are necessary for integrating the equations of disturbed motion.

When the system of parameters Ω, i, p, q, k, u is used, these equations are written as follows:

$$\left.\begin{aligned} \frac{d\Omega}{dt} &= R^{-1} \frac{\sin u}{\sin i} \tilde{W}; \\ \frac{di}{dt} &= R^{-1} \cos u \tilde{W}; \\ \frac{dp}{dt} &= 2pR^{-1} \tilde{T}; \\ \frac{dq}{dt} &= \tilde{S} \sin u + [(q + \cos u) R^{-1} + \cos u] \tilde{T} + k \sin u R^{-1} \operatorname{ctg} i \tilde{W}; \\ \frac{dk}{dt} &= -\tilde{S} \cos u + [(k + \sin u) R^{-1} + \sin u] \tilde{T} - q \sin u R^{-1} \operatorname{ctg} i \tilde{W}; \\ \frac{du}{dt} &= \sqrt{\mu} R^2 p^{-3/2} - R^{-1} \operatorname{ctg} i \sin u \tilde{W}, \\ R &= 1 + q \cos u + k \sin u. \end{aligned}\right\} \tag{6.4}$$

The derivation of the differential equations with respect to Ω, i, p and u is given in [3] (see also [1 or 2]);the derivation of the equations with respect to q and k is given in Appendix V.

The notation used here is: Ω--longitude of the ascending node of the orbit; i--inclination of the orbit (to the plane of the equator); p--focal parameter of the orbit; u--argument of the latitude of the satellite; t--time of motion of the satellite;

$$\tilde{S} = \sqrt{\frac{p}{\mu}}\,S;\quad \tilde{T} = \sqrt{\frac{p}{\mu}}\,T;\quad \tilde{W} = \sqrt{\frac{p}{\mu}}\,W,$$ /60

where S, T and W are the disturbing accelerations directed along the radius, the normal to the radius in the plane of the orbit (the transversal) and the normal to the plane of the orbit (the binormal), respectively.

System of differential equations (6.4) is true for any disturbing function $V(r,\varphi,\lambda)$. Components S, T and W are compared from the formulas:

$$\left.\begin{aligned} S &= \frac{\partial V}{\partial r};\\ T &= \frac{1}{r}\frac{\partial V}{\partial \varphi}\frac{\cos u}{\cos\varphi}\sin i + \frac{1}{r\cos\varphi}\frac{\partial V}{\partial\lambda}\frac{\cos i}{\cos\varphi};\\ W &= \frac{1}{r}\frac{\partial V}{\partial\varphi}\frac{\cos i}{\cos\varphi} - \frac{1}{r\cos\varphi}\frac{\partial V}{\partial\lambda}\frac{\cos u}{\cos\varphi}\sin i. \end{aligned}\right\} \tag{6.4'}$$

In the case where V is a disturbing potential corresponding to one of the models Б, В, Г, Д or Е given in §5, the components of the disturbing acceleration have the following form:

$$S_{\text{Б}} = -\frac{3}{2}c_{20}\,\mu r_0^2\,\frac{1}{r^4}\,(3\sin^2 i\sin^2 u - 1);$$

$$T_{\text{Б}} = 3c_{20}\,\mu r_0^2\,\frac{1}{r^4}\,\sin^2 i\sin^2 u\cos u;$$

$$W_{\text{Б}} = 3c_{20}\,\mu r_0^2\,\frac{1}{r^4}\,\sin i\cos i\sin u;$$

$$S_{\text{В}} = -\frac{5}{8}c_{40}\,\mu r_0^4\,\frac{1}{r^6}\,(35\sin^4 i\sin^4 u - 30\sin^2 i\sin^2 u + 3);$$

$$T_{\text{В}} = \frac{5}{2}c_{40}\,\mu r_0^4\,\frac{1}{r^6}\,\sin^2 i\sin u\cos u(r\sin^2 i\sin^2 u - 3);$$

$$W_{\text{В}} = \frac{5}{2}c_{40}\,\mu r_0^4\,\frac{1}{r^6}\,\sin i\cos i\sin u\,(7\sin^2 i\sin^2 u - 3);$$

$$S_{\Gamma} = -9\mu r_0^2\,\frac{1}{r^4}\,(c_{22}\cos 2\lambda + d_{22}\sin 2\lambda)(1 - \sin^2 i\sin^2 u);$$

$$T_{\Gamma} = 6\mu r_0^2 \frac{1}{r^4}[(d_{22}\cos i - c_{22}\sin^2 i \sin u \cos u)\cos 2\lambda -$$

$$-(d_{22}\sin^2 i \sin u \cos u + c_{22}\cos i)\sin 2\lambda];$$

$$W_{\Gamma} = 6\mu r_0^2 \frac{1}{r^4}[(c_{22}\sin i \cos u - d_{22}\sin i \cos i \sin u)\sin 2\lambda -$$ /61

$$-(c_{22}\sin i \cos i \sin u + d_{22}\sin i \cos u)\cos 2\lambda];$$

$$S_{Д} = -2c_{30}\mu r_0^3 \frac{1}{r^5}\sin i \sin u(5\sin^2 i \sin^2 u - 3);$$

$$T_{Д} = \frac{3}{2}c_{30}\mu r_0^3 \frac{1}{r^5}\sin i \cos u(5\sin^2 i \sin^2 u - 1);$$

$$W_{Д} = \frac{3}{2}c_{30}\mu r_0^3 \frac{1}{r^5}\cos i(5\sin^2 i \sin^2 u - 1);$$

$$S_{Е} = S_{Б} + S_{В} + S_{\Gamma} + S_{Д};$$

$$T_{Е} = T_{Б} + T_{В} + T_{\Gamma} + T_{Д};$$

$$W_{Е} = W_{Б} + W_{В} + W_{\Gamma} + W_{Д}.$$

In the following analysis, the equations of disturbed motion are written with respect to the argument of latitude u. In many cases this formulation is preferable since it permits a geometrically graphic determination of the satellite's period of revolution (draconic period) as the interval of time required for the undisturbed argument of latitude to increase by the quantity 2π[1], and also makes it possible to study an orbit with an extremely low initial eccentricity e_0 (including orbits with $e_0 = 0$). In this case, the use of an angle such as the true anomaly ϑ as the angular argument is inadmissible, since the angle ϑ is reckoned from the perigee of the orbit, which becomes indeterminate when e_0 is small or equal to zero. In addition, since this transition results in the right-hand members of the equations of motion becoming an explicit function of the argument, integration of the system is facilitated (see Chapter Three). Transition to the argument u is accomplished by multiplying the right-hand and left-hand members of the first five equations in system (6.4) respectively, by the quantity (see [13]):

$$\frac{dt}{du} = \sqrt{\frac{p}{\mu}}\, p R^{-2} + \frac{p^3}{\mu} R^{-5} \cot i \sin u \tilde{W} \tag{6.5}$$

[1] Strictly speaking, the draconic period is defined as the time of motion between two consecutive transits of the ascending node of the orbit [19]. In non-Soviet literature, this period is called the *nodal* period [47]; [47] also gives a comparative analysis of the various relationships derived by non-Soviet authors for determining the disturbed draconic period.

And by substituting equation (6.5) for the sixth equation in system (6.4). /62

In studying disturbances in the field of the various models, the gravitational potential, as was pointed out above, is a function of a definite negative order of magnitude with respect to polar flattening of the Earth. Components S, T and W will also have the same negative order of magnitude. After transition to the argument u, products of functions $\tilde{W}W$, $\tilde{W}S$ and $\tilde{W}T$ will appear in the right-hand members of the differential equations, and as a result, the disturbing functions will contain terms with a higher negative order of magnitude than the disturbing potential. This should be kept in mind in proper formulation of the problem.

Specifically, with an accuracy to terms of the order of the square of polar flattening of the Earth (inclusive), the system of equations with respect to argument u is written in the form [48]:

$$\left.\begin{aligned}
\frac{d\Omega}{du} &= \frac{p^{3/2}}{\sqrt{\mu}} R^{-3}\frac{\sin u}{\sin i}[\tilde{W}_i+\theta_1];\\
\frac{di}{du} &= \frac{p^{3/2}}{\sqrt{\mu}} R^{-3}\cos u[\tilde{W}_i+\theta_1];\\
\frac{dp}{du} &= \frac{2p^{5/2}}{\sqrt{\mu}} R^{-3}[\tilde{T}_i+\theta_2];\\
\frac{dq}{du} &= \left(-q\sin^2 u+\frac{1}{2}k\sin 2u\right)\cot i\frac{di}{du}+\\
&+\left(k\sin^2 u-\frac{1}{2}q\sin 2u\right)\cos i\frac{d\Omega}{du}+\frac{1}{2p}(q+\cos u)\frac{dp}{du}+\\
&+\frac{p^{3/2}}{\sqrt{\mu}}R^{-2}[\tilde{S}_i\sin u+\tilde{T}_i\cos u+\theta_3];\\
\frac{dk}{du} &= \left(-k\sin^2 u-\frac{1}{2}q\sin 2u\right)\cot i\frac{di}{du}-\\
&-\left(-q\sin^2 u+\frac{1}{2}k\sin 2u\right)\cos i\frac{d\Omega}{du}+\frac{1}{2p}(k+\sin u)\frac{dp}{du}+\\
&+\frac{p^{3/2}}{\sqrt{\mu}}R^{-2}[\tilde{T}_i\sin u-\tilde{S}_i\cos u+\theta_4];\\
\frac{dt}{du} &= \sqrt{\frac{p}{\mu}}\,p R^{-2}+\frac{p^3}{\mu}R^{-5}\cot i\sin u\tilde{W}_i,\\
R &= 1+q\cos u+k\sin u.
\end{aligned}\right\}\quad (6.6)$$

The subscript i in functions $\tilde{S}_i$, $\tilde{T}_i$, $\tilde{W}_i$ assumes the meanings Б, В, Г, Д, Е, /63
i.e., S, T and W contain all terms of the projection of the disturbing force corresponding to the given model of the potential field (beginning with terms of the first negative order of magnitude).

Functions θ_1, θ_2, θ_3 and θ_4 are quantities of the second negative order of magnitude with respect to flattening of the Earth, and are defined as follows (the subscript 1 associated with components $\tilde{S}_1$, $\tilde{T}_1$ and $\tilde{W}_1$ indicates that these components contain terms of only the first negative order of magnitude):

$$\theta_1 = \frac{p^2}{\mu} R^{-3} \cot i \sin u \tilde{W}_1^2;$$

$$\theta_2 = \frac{p^2}{\mu} R^{-3} \cot i \sin u \tilde{W}_1 \tilde{T}_1;$$

$$\theta_3 = \theta_3^* \sin u + \theta_2 \cos u;$$

$$\theta_4 = \theta_2 \sin u - \theta_3^* \cos u,$$

where

$$\theta_3^* = \frac{p^2}{\mu} R^{-3} \cot i \sin u \tilde{W}_1 \tilde{S}_1.$$

After substituting the values of $\tilde{S}_1$, $\tilde{T}_1$ and $\tilde{W}_1$, we get:

$$\theta_1 = 9c_{20}^2 \frac{r_0^4}{p^5} R^5 \sin i \cos^3 i \sin^3 u;$$

$$\theta_2 = 9c_{20}^2 \frac{r_0^4}{p^5} R^5 \sin i \cos^2 i \sin^3 u \cos u;$$

$$\theta_3 = 9c_{20}^2 \frac{r_0^4}{p^5} R^5 \cos^2 i \sin^3 u [\sin i \cos^2 u - \tfrac{1}{2}(3\sin^2 i \sin^2 u - 1)];$$

$$\theta_4 = 9c_{20}^2 \frac{r_0^4}{p^5} R^5 \cos^2 i \sin^2 u \cos u [\sin i \sin^2 u (3\sin^2 i + 1) - 1].$$

If the model of the gravitational field is taken as Г,E or some other model which describes the asymmetric nature of the Earth with respect to its axis of rotation, terms will appear in the right-hand members of equations (6.4) which depend on the satellite's longitude λ, and the system of differential equations should then be supplemented by an expression which takes account of the effect which the Earth's rotation and precession of the plane have on /64
the amplitude of the disturbances:

$$\lambda = \Omega + \lambda^* - [\lambda^\circ + \omega_{\text{Earth}} (t-t_0)]. \qquad (6.6')$$

The quantity λ here is the instantaneous longitude of the satellite reckoned from the point of the vernal equinox (the point γ); t is the instantaneous time of motion of the satellite; t_0 is the initial instant; λ^0 is the longitude of the Greenwich meridian with respect to point γ at time t_0; ω_{Earth} is the angular rotation of the Earth; Ω is the instantaneous value of the longitude of the ascending node.

The quantity λ^* is the change in longitude of the satellite (with respect to the ascending node) resulting from its orbital motion:

$$\sin\lambda^* = \cos i \sin u / \sqrt{1-\sin^2 i \sin^2 u}.$$

The derivation of this relationship, as well as the equation for determining λ will be apparent from Fig. 2. Here, the following relationships

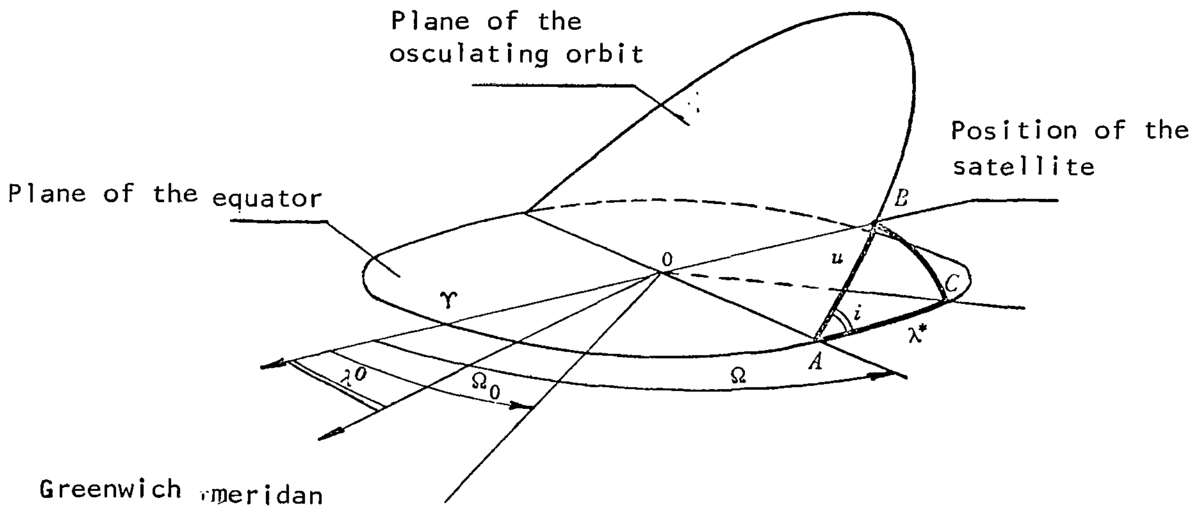


Figure 2.

from spherical trigonometry are used:

$\sin\lambda^* = \sin u \sin B$ (law of sines for right triangle ABC);

$\sin B = \cos i/\cos\varphi$ (law of cosines for the same triangle);

$$\sin B = \cos i/\sqrt{1-\sin^2\varphi} = \cos i/\sqrt{1-\sin^2 i\sin^2 u}.$$

Determination of the angle λ^* from the expression for arcsin λ^* should /65 not cause any difficulties with machine computation. The initial value of λ^* is uniquely determined according to the known position of the satellite. This angle may be determined in the course of the computations uniquely from its preceding value.

Perturbation of the osculating orbit is characterized by the five quantities

$$\delta\Omega=\Omega-\Omega_0,\ \delta i=i-i_0,\ \delta p=p-p_0,\ \delta q=q-q_0,\ \delta k=k-k_0, \tag{6.7}$$

where the functions without subscripts are the instantaneous values, while those with a subscript are the initial values of the osculating elements.

By varying the known equations which relate the rectangular inertial geocentric coordinates to Kepler's elements [3] (see also [1 or 2]):

$$\left.\begin{aligned} x &= r(\cos u\cos\Omega - \sin u\sin\Omega\cos i);\\ y &= r(\cos u\sin\Omega + \sin u\cos\Omega\cdot\cos i);\\ z &= r\sin u\sin i, \end{aligned}\right\} \tag{6.8'}$$

we get the perturbations of the rectangular geocentric coordinates as functions of the perturbations of the osculating parameters:

$$\left.\begin{aligned} \delta x &= (\cos u\cos\Omega - \sin u\sin\Omega\cos i)\,\delta r - r(\cos u\sin\Omega +\\ &\quad + \sin u\cos\Omega\cos i)\,\delta\Omega + r\sin u\sin\Omega\sin i\,\delta i;\\ \delta y &= (\cos u\sin\Omega + \sin u\cos\Omega\cos i)\,\delta r + r(\cos u\cos\Omega -\\ &\quad - \sin u\sin\Omega\cos i)\,\delta\Omega - r\sin u\cos\Omega\sin i\,\delta i;\\ \delta z &= \sin u\sin i\,\delta r + r\sin u\cos i\,\delta i. \end{aligned}\right\} \tag{6.8}$$

Here δr is the perturbation of the absolute value of the radius vector:

$$\delta r = |\vec{r}| - |\vec{r}_H| = r_H\left[\frac{\delta p}{p} - R^{-1}(\cos u\,\delta q + \sin u\,\delta k)\right], \tag{6.9}$$

where r_H are the values of the radius during motion in a normal field for the same value of the argument u.

A graphic characteristic of the disturbed potential of the satellite is the quantity $\Delta r = |\vec{r} - r_H|$. Since

$$\Delta r = \sqrt{(x-x_H)^2 + (y-y_H)^2 + (z-z_H)^2},$$

by substituting the quantities $x-x_H=\delta x, y-y_H=\delta y, z-z_H=\delta z$, , we get Δr as a function of the disturbances of the Keplerian elements of the orbit:

$$\Delta r = [\delta r^2 + r_H^2(\cos^2 u + \cos^2 i \sin^2 u)\delta\Omega^2 + r_H^2 \sin^2 u \delta i^2 - 2r_H^2 \sin u \cos u \sin i \delta i \delta\Omega]^{1/2}. \quad (6.10)$$ /66

Functions (6.8), (6.9) and (6.10) are derived on the assumption that the positions of the "disturbed" and "undisturbed" satellites (i.e. the positions of the satellites during disturbed and undisturbed motion) are compared for the same value of the argument of the latitude. Thus, the argument of the motion is the angle u (which corresponds to formulating the system of equations in form (6.6)). This motion may be called isogonal as distinct from isochronous motion in which the argument is time (system (6.4)).

In deriving the expressions of δx, δy, δz, δr and Δr for isochronous motion, it would be necessary to take account of the relationship $u = u(t)$ and to take the partial differential of the function u with respect to t everywhere.

This derivation is not given here since we shall only be considering isogonal disturbed motion. This motion may be completely characterized by the two functions:

$$\Delta r \text{ and } \delta t = t - t_H. \quad (6.11)$$

The quantity δt is time perturbation. Thus, while function (6.10) shows that the disturbed satellite will be located at some point on the surface of a sphere (of radius Δr) whose center coincides with the undisturbed satellite (the position of this satellite on the surface is not defined since we are considering the modulus of the vectoral difference in (6.10)), function

(6.11) shows that the satellite will arrive at this point with a time difference of δt with respect to the instant when the undisturbed satellite will be at the center of the given sphere.

Functions (6.10) and (6.11) give information only on the amplitude of the disturbances (while functions (6.7) or (6.8) describe their geometry), therefore, in considering this particular characteristic of motion we may say approximately that the disturbed satellite at instant t_H corresponding to a given value of the argument of latitude $u = u^*$ stays within a tube of radius Δr and angular length δu. The quantity δu corresponds to the time δt of motion along the orbit, while both "ends" of the tube are convex hemispheres of radius Δr (Fig. 3). The region found in this way gives a graphic evaluation of the amplitude of disturbances in the motion of the satellite.

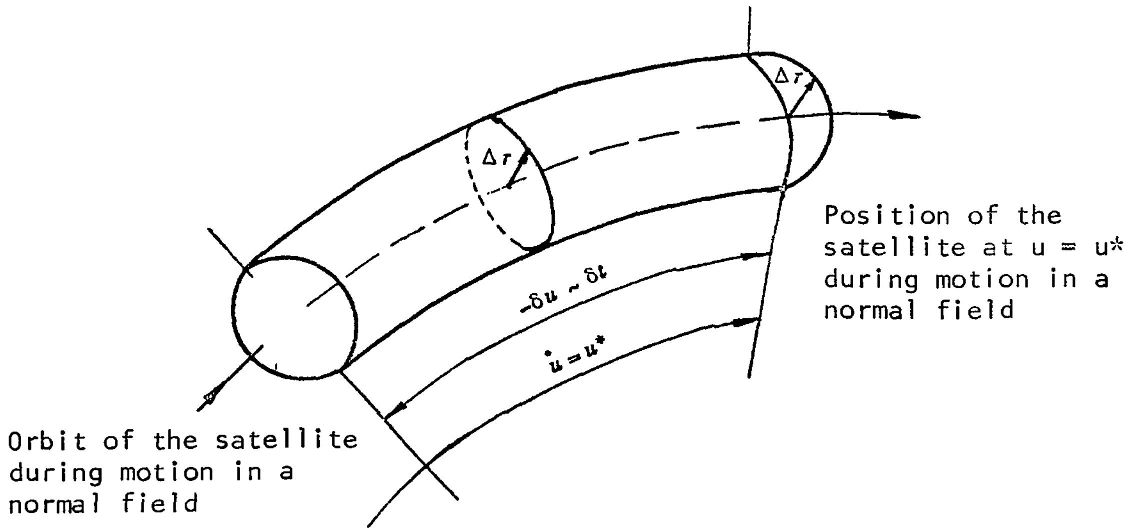


Figure 3 /67

Perturbations of the orbital elements are qualitatively different. Since the right-hand members of equations (6.4) and (6.6) are periodic functions of the argument, it is obvious that the functions Ω, i, p, q, k, t and u must contain a periodic component with a period equal to the time of revolution of the satellite. The nonlinearity of the right-hand members of the equations with respect to the periodic terms indicates that these functions represent a superposition of harmonic components, and must also contain terms which are proportional to the various powers of the argument.

The so-called secular terms in the solutions of the equations (in celestial mechanics they are called inequalities), increase with an increase in the argument[1]. The short-period disturbances have a period equal to the period of revolution of the given heavenly body. The long-period disturbances have a period greater than the period of revolution [1, 2].

Long-period disturbances appear when there is a change in the right-hand members of the differential equations (disturbing functions) with a period greater than the period of revolution of the moving body. In the given case of satellite motion (the disturbing function is only the eccentricity of the Earth's field) the long-period disturbances are the consequence of two factors: the diurnal rotation of the Earth (if the asymmetry of the Earth with respect to the axis of rotation is considered, as for instance in models Г and Е) and rotation of the line of apsides which is directly due to a change in functions q and k with a period of more than 2π with respect to argument u. /68

If satellite motion is considered on a comparatively short interval (as for instance in the discussion below, where this interval is taken as less than one day), then the long-period disturbances do not have time to show their periodic nature, and show no difference from secular terms in analysis of computational data. Therefore, it is advisable in these cases to speak of quasi-secular disturbances (meaning that they contain both long-period and secular components) and periodic disturbances (i.e. disturbances with a short-period).

In general however, the problem of the presence of secular disturbances in some orbital element or another is extremely complex and may be analytically solved only to a certain approximation. At the same time, this problem is of definite interest since it characterizes the stability of satellite orbits (see the papers by V. G. Demin [49, 50].

§7. Disturbances in the Elements of Orbits in the Central Gravitational Field of the Earth

Over short intervals of satellite motion, the disturbances in the orbital elements which are greatest in amplitude are periodic in nature. This is due to the fact that the secular terms (see footnote in §6) in the solutions for the equations of motion show up because of terms of a higher negative order of magnitude which are present in the expansion of the potential. These terms may have an appreciable effect on motion only over a long time interval. Periodic disturbances, on the other hand, are determined by the first terms of the expansion which have coefficients with a higher absolute value.

---

[1] In conformity with the functions $u^n[A\cos^{m_1}(\nu_1 u+c_1)+B\sin^{m_2}(\nu_2 u+c_2)]$, which appear in the solutions of the given equations, they are sometimes called secular terms. In celestial mechanics, such perturbations are also sometimes called mixed terms. The quantities n, m_1, m_2, ν_1, ν_2, c_1, c_2, A and B here are constants.

Periodic Disturbances

The nature and amplitude of periodic disturbances in the elements are determined by the effect of the second zonal harmonic in the expansion of the /69 gravitational potential. Thus, motion in the field of model Б is the determining factor of the qualitative and quantitative picture in all cases.

To make our analysis specific, let us assume in further discussion that $u_0 = 0$, i.e. that the initial position of the satellite is over the equator (at the ascending node of the orbit).

The periodic and secular disturbances of the orbital elements during satellite motion in various models of the gravitational field are shown in Figures 4-28. The corresponding variants are shown in Table 5.

TABLE 5 *

Variant Number	Initial Conditions					Altitude	
	Ω_0	i_0	p_0, км	e_0	ω_0	Apogee h_A, км	Perigee h_Π, км
1	0	90°	6996,09	0,0499	0	1000	300
2	0	63°26′	6996,09	0,0499	0	1000	300
3	0	63°26′	7363,55	0	–	1000	1000
4	0	10°	7363,55	0	–	1000	1000
5	0	45°	6996,09	1,010	0	–	1000

The inclination of the plane i and focal parameter p (Fig. 4, 10) show the simplest change, close to harmonic law, in model Б. Regardless of the values of the initial parameters of the orbit, fluctuations in these elements have two equal maxima at values of u equal to $\pi/2$ and $3\pi/2$.

Considerably more complex in form are disturbances in the angular distance of the perigee ω, eccentricity e and components q and k of the Laplace vector (Figures 5-8).

Clearly evident in the perturbation of the longitude Ω of the ascending node is the secular component which increases linearly with an increase in angle u. Superimposed on this term is a harmonic component (Fig. 9). Similar, but more complex in nature, is disturbance of the function $t_\Omega(u)$. The longitude of the ascending node Ω and inclination i do not have periodic terms for

* Tr. Note: Commas indicate decimal points.

initial values $i_o = \pi/2$, i.e. for polar orbits.

The function $\delta r(u)$ has a more or less pronounced extremum in all instances which is reached in the region close to π (Figures 12 and 13).

Typically, at values of $0 \leq i_0 \leq \pi/2$, periodic perturbations $\delta\Omega(u)$, $\delta i(u)$, $\delta p(u)$, $\delta t(u)$ and $\delta r(u)$ are nearly always negative.

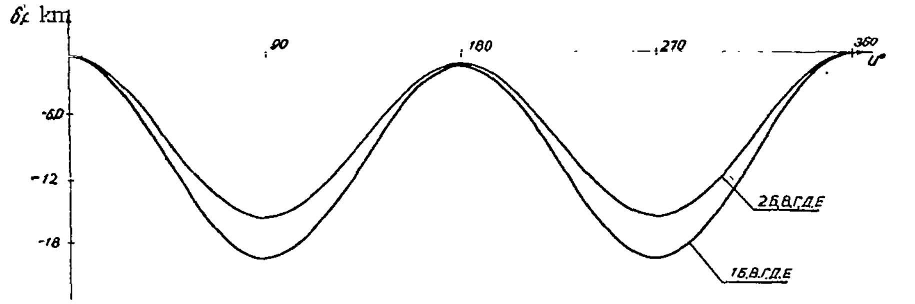


Fig. 4.

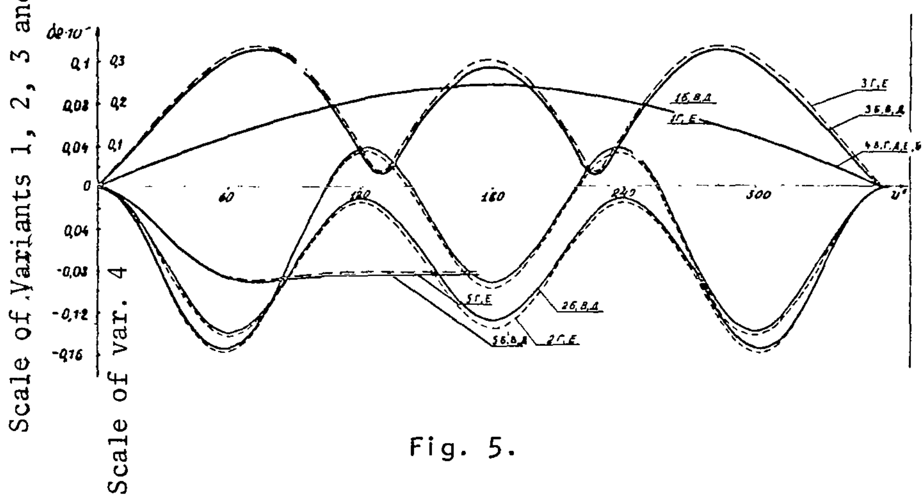


Fig. 5.

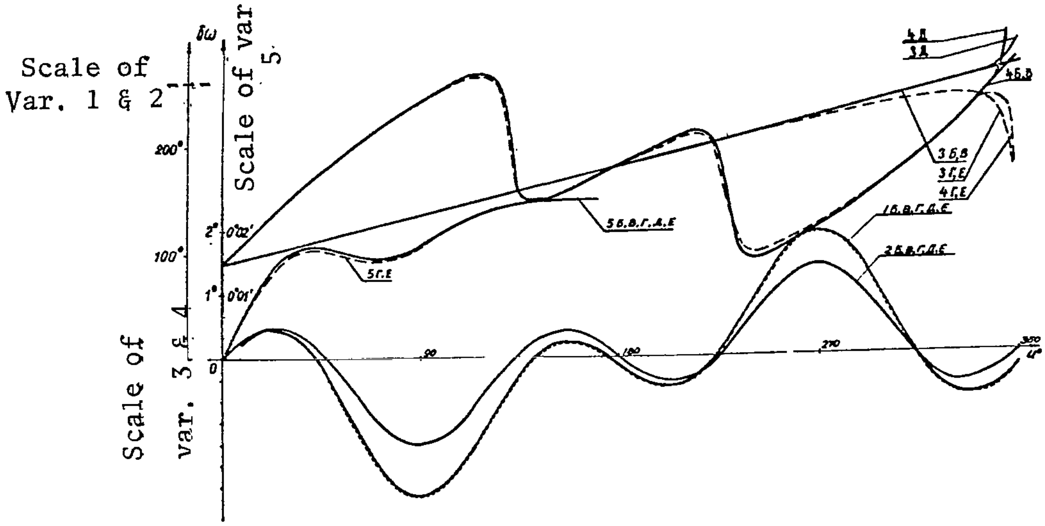


Fig. 6

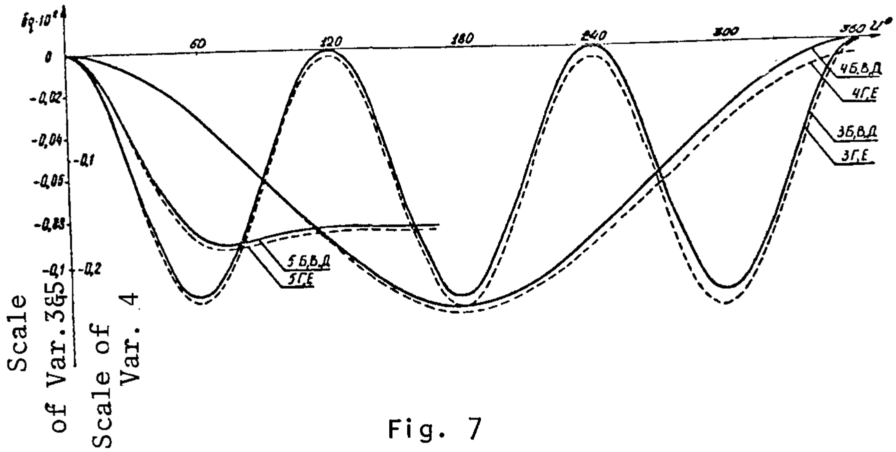


Fig. 7

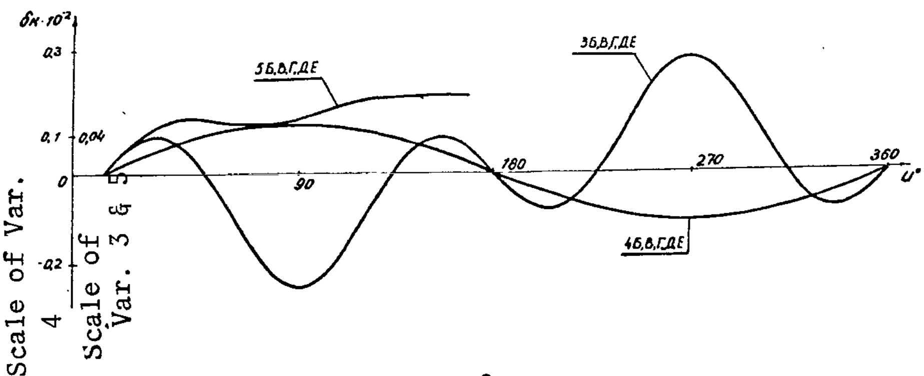


Fig. 8

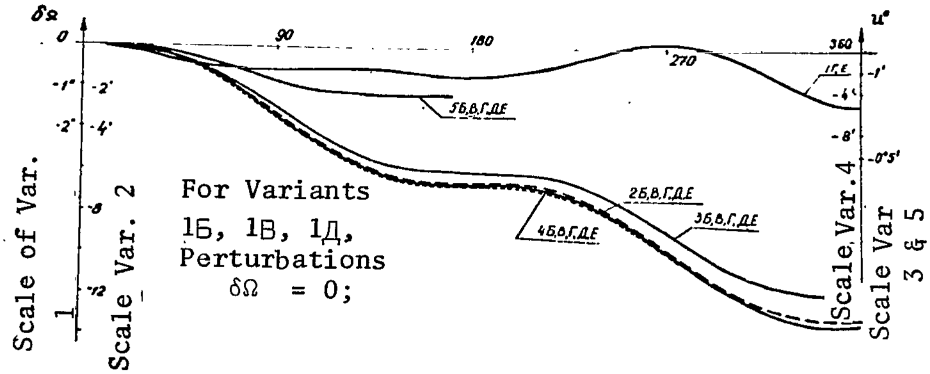


Fig. 9.

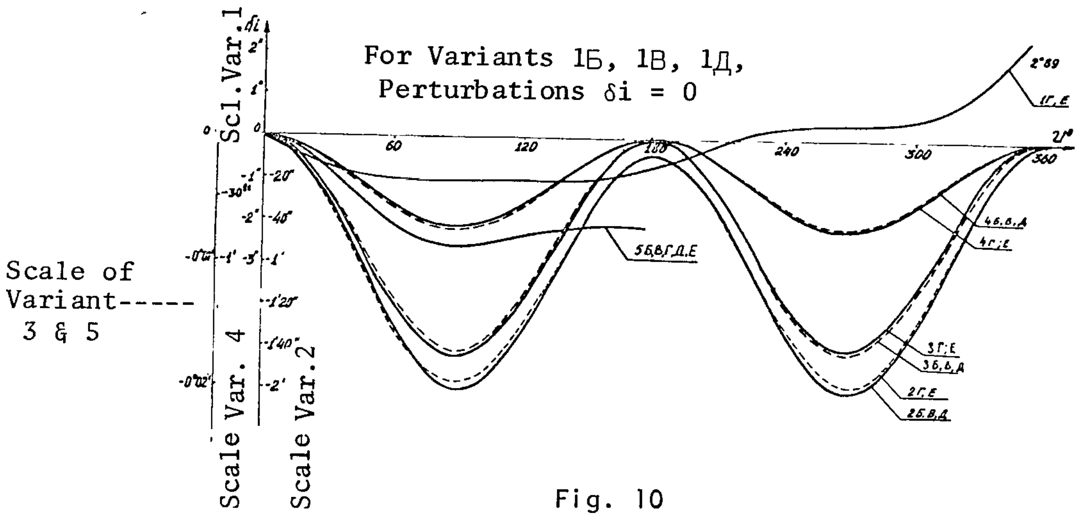


Fig. 10

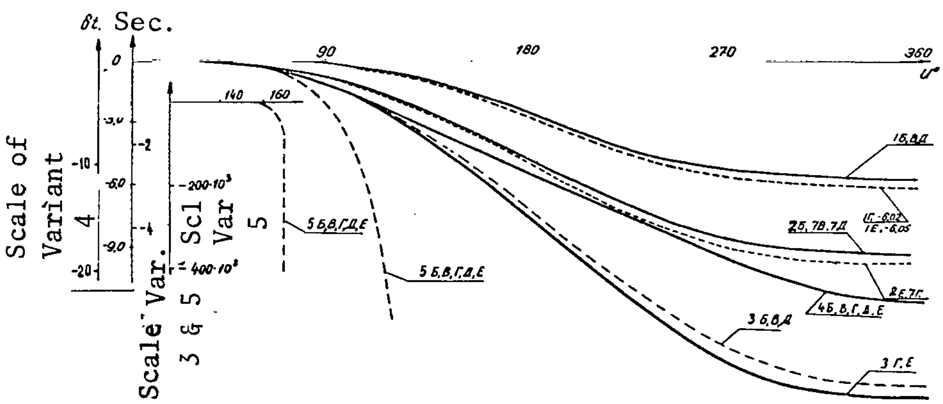


Fig. 11

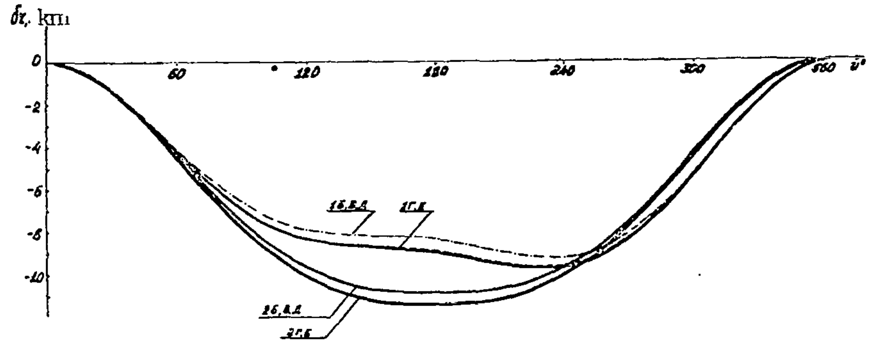


Fig. 12

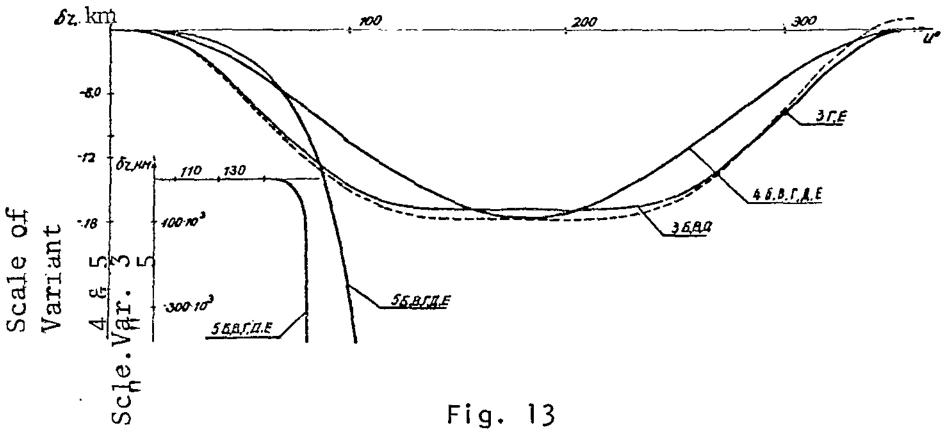


Fig. 13

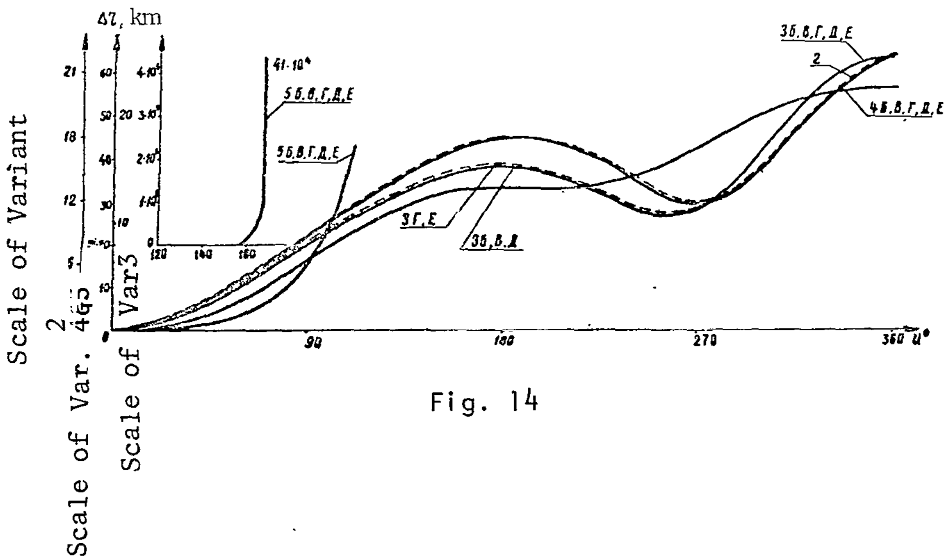


Fig. 14

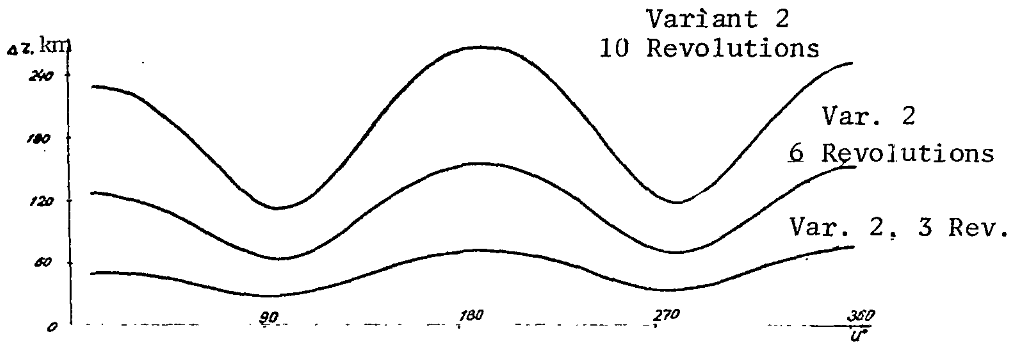


Fig. 15

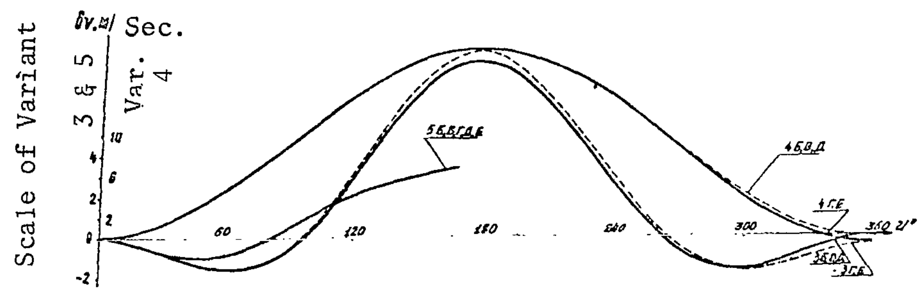


Fig. 16

The nature of the change in the quantity $\Delta r(u)$ is easily explained by the form of this function, which depends on the sum of δr^2, $\delta\Omega^2$, and δi^2. Therefore, when $i_0 = \pi/2$ (in this case $\delta\Omega = \delta i = 0$), the graph of Δr repeats the function δr reflected with respect to the horizontal axis. In the remaining instances the function $\delta r(u)$ has a maximum at $u = 2\pi$ corresponding to the maximum of function $\delta\Omega(u)$. /74

Up until now, we have been speaking of perturbations in elliptical orbits. The relationship between perturbations and initial conditions (including the initial value of eccentricity e_0) is taken up, generally speaking later on; however, even now we can mention some characteristic singularities in the perturbation of circular and hyperbolic orbits[1].

It should be noted first of all, that in the case of initial conditions corresponding to circular motion, an osculating orbit will be elliptical; related disturbances $\delta e(u)$ will be only positive (see Fig. 5). Perturbations of the line of apsides are quite typical: at certain values of the initial

[1] Rather than the disturbance of circular orbits, it would be more accurate to speak of perturbation of the orbits of circular satellites in the same sense considered in Appendix VI.

inclination (e.g. at $i_0 = 90°$), it rotates at an angular velocity greater than the rate of satellite revolution, while at other inclinations (e.g. $i_0 = 63.4°$), it oscillates.

As distinct from the function $\delta\omega(u)$ which has a discontinuity within the interval $[0,2\pi]$, the functions $\delta q(u)$ and $\delta k(u)$ are continuous. This justifies the use of these functions in studying the disturbances of orbital elements of satellites.

The problem of disturbances of circular orbits is taken up in more detail in Appendix VI. The approximate solutions found in the third chapter are used there.

The periodic perturbations of hyperbolic orbits are basically the same in nature as those of elliptical orbits (see Fig. 5-11). The difference shows up in the extremely rapid (with respect to argument u) increase in the functions $\delta t(u)$ and $\delta r(u)$ (and consequently in $\Delta r(u)$). This situation is explained in §8. It should be kept in mind that values of the focal radius of about 900,000 - 1,000,000 km correspond to values of the argument u = 170° for which numerical values are given in graphs and tables. Besides, perturbations in the eccentricity of hyperbolic and parabolic orbits (see Fig. 5) are typically always negative.

TABLE 6*

A	δp, km $u = 90°$		$\delta e \cdot 10^2$; $u = 60°$		$\delta\omega$; $u = 90°$		$\delta\Omega$; $u = 360°$		δi		δt^S; $u = 360°$		Δr_{max}, km	
	1	2	1	2	1	2	1	2	1	2	1	2	1	2
Б	−19,71	−15,76	−0,1554	−0,1395	−2°10′33″	−1°22′38″	0	−13′12″	0	−1′56″	−5,62	−9,23	9,266	25,293
В	−19,68	−15,77	−0,1556	−0,1397	−2°10′12″	−1°22′37″	0	−13′10″	0	−1′56″	−5,65	−9,25	9,303	25,230
Г	−19,92	−16,02	−0,1587	−0,1430	−2° 9′53″	−1°22′26″	−1″,33	−13′21″	2″,89	−2′00″	−6,02	−9,65	9,688	25,581
Д	−19,66	−15,74	−0,1557	−0,1394	−2°10′ 8″	−1°22′33″	0	−13′13″	0	−1′56″	−5,62	−9,24	9,235	25,293
Е	−19,83	−16,00	−0,1588	−0,1431	−2° 9′ 7″	−1°22′21″	−1″,33	−13′19″	2″,89	−2′00″	−6,05	−9,66	9,745	25,507

Key A = Model of Gravitational Field

Note: Given in column 1 are disturbances for an orbit with the following parameters: $\Omega_0 = 0$; $i_0 - \pi/2$; $\omega_0 = 0$; $e_0 = 0.0499$; $p_0 = 6,996$ km (altitude ot the apogee $h_A = 1,000$ km, altitude of the perigee $h_\Pi = 300$ km). Given in column 2 are the disturbances for an orbit with the following parameters: $\Omega_0 = 0$; $i_0 = 63°26'$; $\omega_0 = 0$; $e_0 = 0.499$; $p_0 = 6,996$ km.

*Tr. Note: Commas indicate decimal points.

TABLE 7 *

A	δp,		δe · 10²		δω		δΩ		δi; u = 90°		δt^S		δr, км		Δr, км		δV, м/сек	
	u = 90°	u = 90°	u = 180°	u = 120°	u = 120°	u = 170°	u = 360°	u = 170°	u = 90°	u = 90°	u = 360°	u = 170°	u = 180°	u = 170°	u = 360°	u = 170°	u = 180°	u = 170°
	1	2	1	2	1	2	1	2	1	2	1	2	1	2	1	2	1	2
Б	−14,48	−7,54	0,116	−0,0820	180°	2′28″	−11′56″	−2′20″	−1′42″	−42″	−7,74	−428500	−8,45	−412900	25,56	412900	8,45	3,31
В	−14,49	−7,56	0,115	−0,0822	179°58′1″	2′28″	−11′54″	−2′20″	−1′42″	−42″	−7,74	−428300	−8,46	−413400	25,49	413400	8,46	3,31
Г	−14,73	−7,81	0,121	−0,0848	175°52′3″	2′29″	−11′58″	−2′20″	−1′44″	−43″	−7,98	−439100	−8,89	−423400	25,59	423400	8,89	3,32
Д	−14,46	−7,55	0,115	−0,0820	179°51′59″	2′28″	−11′56″	−2′20″	−1′42″	−42″	−7,74	−428300	−8,45	−412700	25,55	412700	8,46	3,29
Е	−14,72	−7,82	0,121	−0,0847	175°46′30″	2′29″	−11′55″	−2′20″	−1′44″	−43″	−7,98	−438700	−8,89	−423000	25,52	423000	8,91	3,31

Key A = Model of Gravitational Field

Note: Disturbance $\delta\omega$ in column 1 was calculated with respect to the initial (in the sense $u \to 0$) value $\omega = 90°$ (see Appendix VI). Given in column 1 are the disturbances for an orbit with the following parameters: $\Omega_0 = 0$; $i_0 = 63°24'$; $e_0 = 0$; $h_A = h_\Pi = 1{,}000$ km. Given in column 2 are the disturbances for an orbit with the following parameters: $\Omega_0 = 0$; $i_0 = 45°$; $\omega_0 = 0$; $e_0 = 1.010$; $p_0 = 14{,}800$ km; $h_\Pi = 1{,}000$ km.

The numerical values of the periodic disturbances of elliptical, circular and hyperbolic orbits are given in Tables 6 and 7, and are also apparent from the graphs. /76

It may be noted that the absolute values of perturbations $\delta r(u)$, $\Delta r(u)$ and $\delta t(u)$ increase in the field of model Б with an increase in eccentricity to values greater than or equal to 1. On the other hand, disturbances in the orbital velocity are somewhat greater for circular orbits and less for hyperbolic orbits.

Periodic disturbances in elliptical and circular orbits in the field of other models (В, Г, Д, Е) differ very slightly from those in the field of model Б. Exceptions to this rule take place only for the inclination of the plane and the longitude of the ascending node of polar orbits, which are disturbed only in models which take account of the triaxial shape of the Earth (see Fig. 9 and 10). It may also be noted that the elements p, e and $\Delta r(u)$ in the field of an asymmetric spheroid (model Д) are disturbed as strongly as in the field of a triaxial ellipsoid (model Г).

The differences in the values of functions $\Delta r(u)$ and $\delta t(u)$ for hyperbolic orbits in the field of various models reach extremely high values. For

*Tr. Note: Commas indicate decimal points.

instance for an orbit with an eccentricity $e = 1.01$ at $u = 170°$, these differences are equal to 10,300 km and $10{,}800^s$ for Δr and δt respectively, (of course, the space vehicle in this case is about 1 million km from the Earth). This shows that an optimum approximation to the field of the geoid must be represented by the field of the selected model when calculating motions along hyperbolic trajectories over a comparatively large interval of variation in the argument.

Quasisecular Disturbances

The quasisecular disturbances of all elements in the field of models Б В or Д are linear (nearly linear) or equal to zero.

For instance, the focal parameter (see Fig. 17) in the field of models Б and В undergoes slight negative perturbations (for a polar orbit with $h_A = 1{,}000$ km, $h_\Pi = 300$ km, δp is no more than 0.5 meter on the tenth revolution), and only in a field which takes account of the Earth's asymmetry (model Д) do these disturbances become more appreciable. For orbits with inclination $i_0 = i_0^* = 63.4°$, perturbation δp disappear in the first two cases (Б and В) and are extremely slight in the third (Д).

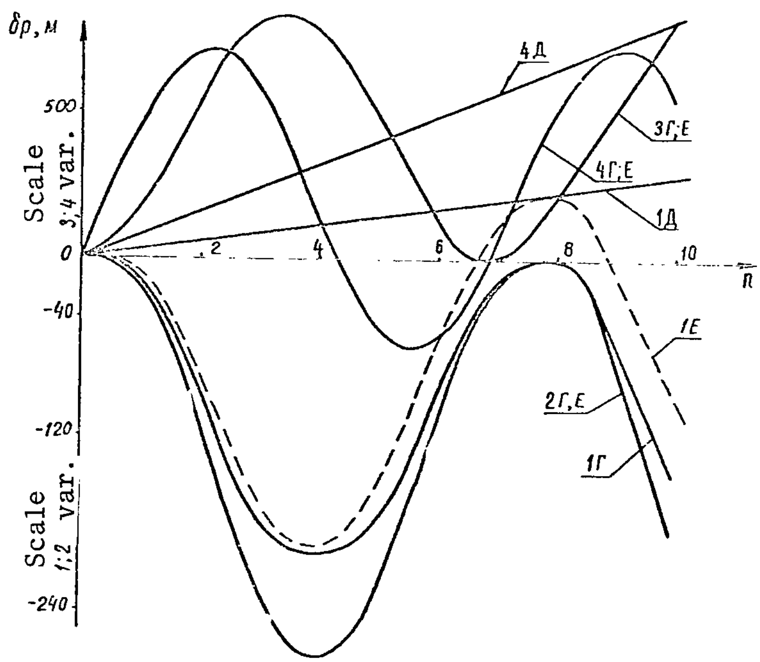
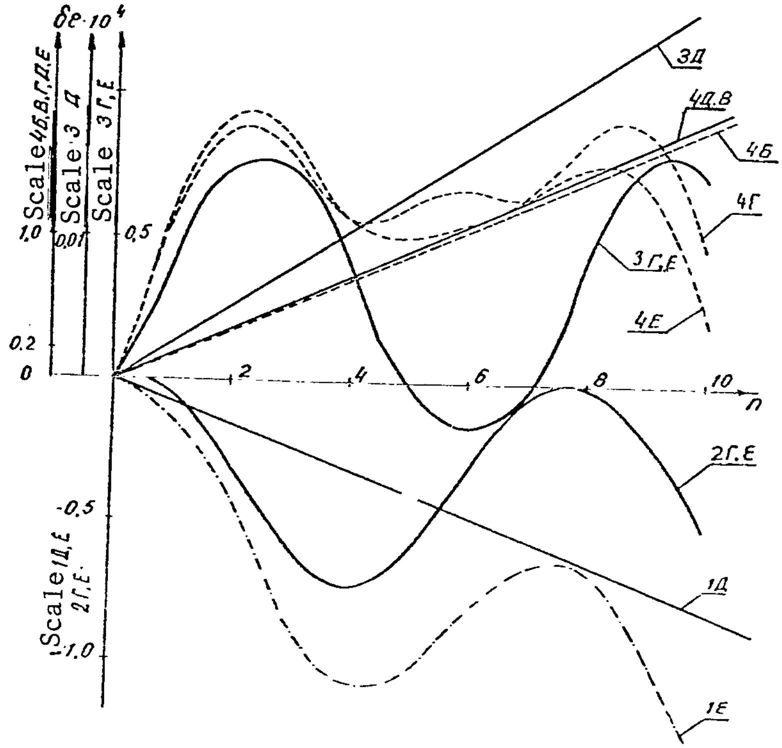
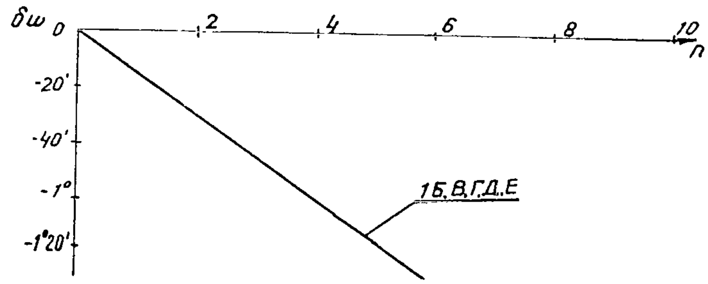


Fig. 17



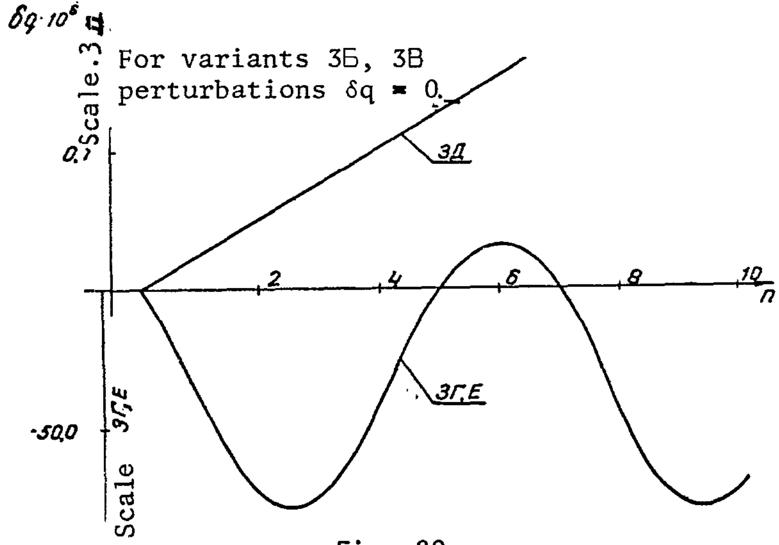
δε·10⁴
Scale 4Б,В,Г,Д,Е
Scale 3 Д
Scale 3Г,Е
1,0
0,01
0,5
0.2
0
2
4
6
8
10
n
3Д
4Д,В
4Б
4Г
3Г,Е
4Е
2Г,Е
1Д
1Е
-0,5
Scale 1Д,Е
2Г,Е
-1,0

Fig. 18



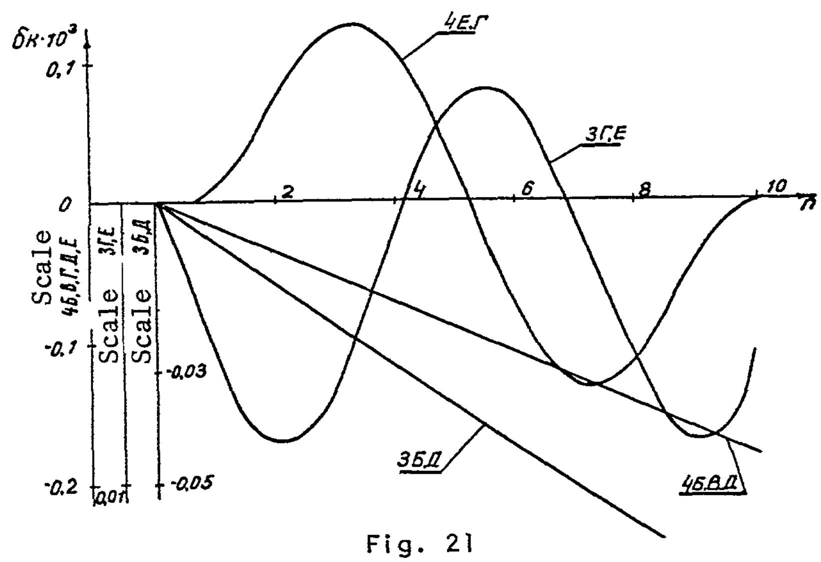
δω 0
2
4
6
8
10
n
-20'
-40'
-1°
-1°20'
1Б,В,Г,Д,Е

Fig. 19



δq·10⁶
Scale 3Д
For variants 3Б, 3В
perturbations δq = 0.
0.1
3Д
2
4
6
8
10
n
3Г,Е
-50,0
Scale 3Г,Е

Fig. 20



δк·10³
0,1
4Е,Г
3Г,Е
0
2
4
6
8
10
n
Scale 4Б,В,Г,Д,Е
Scale 3Г,Е
Scale 3Б,Д
-0,1
-0,03
3БД
4БВД
-0,2
0,01
-0,05

Fig. 21

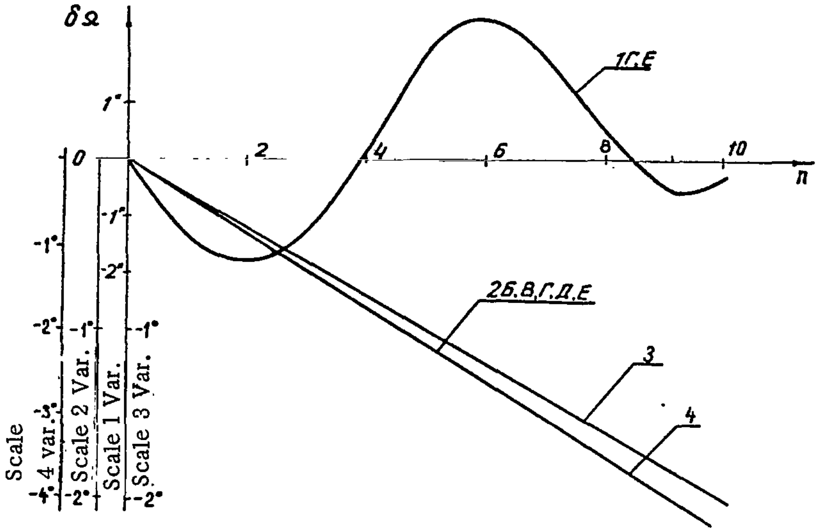

δΩ
1°
0
-1°
-2°
-3°
-4°
Scale 4 var.
Scale 2 Var.
Scale 1 Var.
Scale 3 Var.
2
4
6
8
10
n
1Г,Е
2Б,В,Г,Д,Е
3
4

Fig. 22

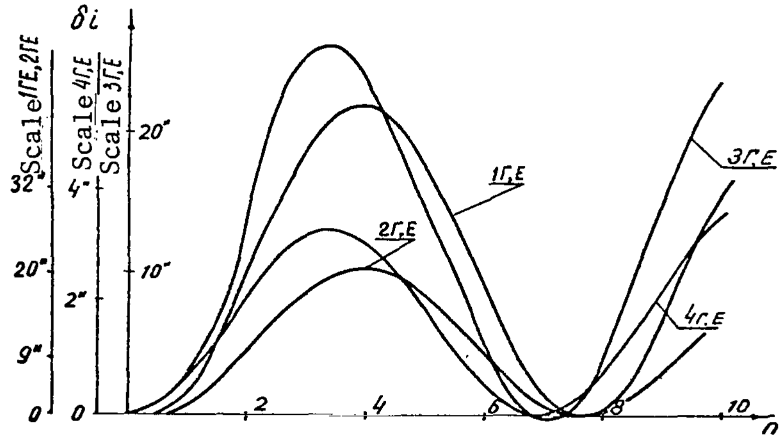

δi
Scale 1ГЕ,2ГЕ
Scale 4Г,Е
Scale 3Г,Е
20"
10"
32"
20"
9"
4"
2"
0
2
4
6
8
10
n
1Г,Е
2Г,Е
3Г,Е
4Г,Е

Fig. 23

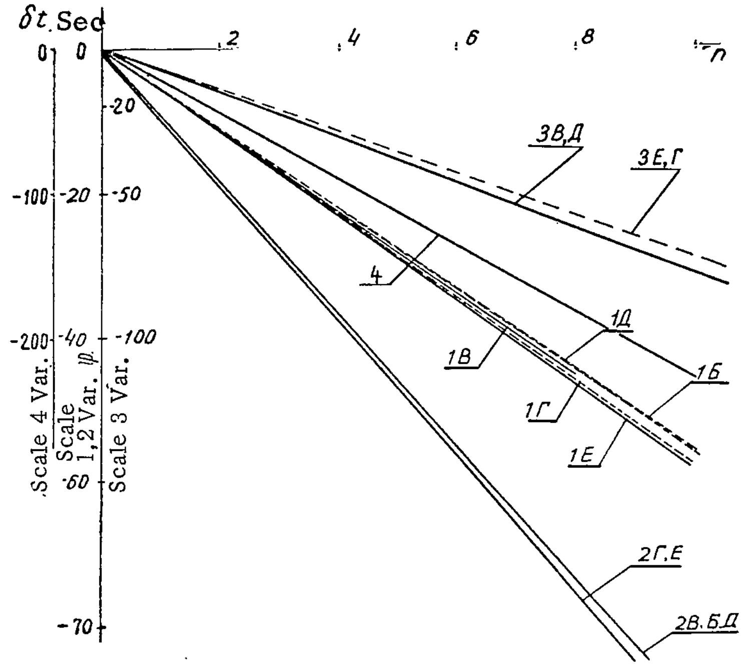

δt, Sec
0
-20
-50
-100
-200
-40
-60
-70
Scale 4 Var.
Scale 1,2 Var.
Scale 3 Var.
2
4
6
8
n
3В,Д
3Е,Г
4
1Д
1В
1Б
1Г
1Е
2Г,Е
2В,Б,Д

Fig. 24

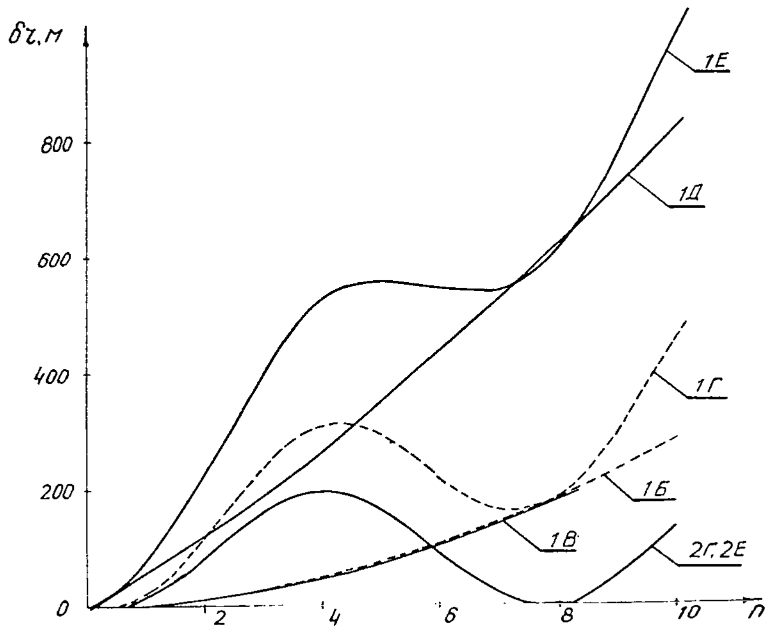

δz, м
800
600
400
200
0
2
4
6
8
10
n
1Е
1Д
1Г
1Б
1В
2Г,2Е

Fig. 25

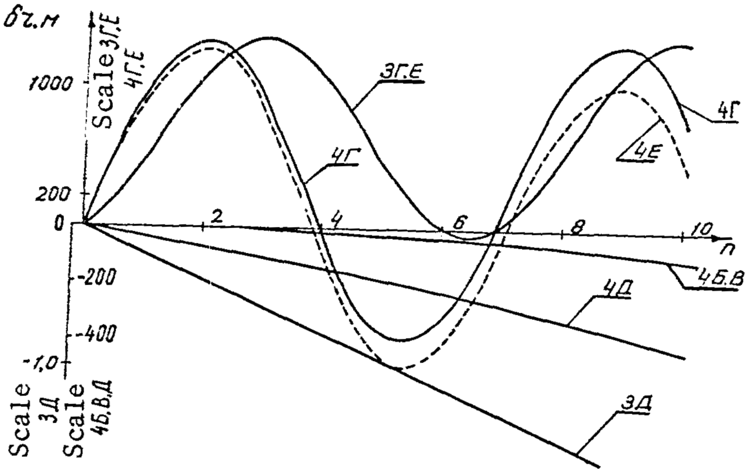


Fig. 26

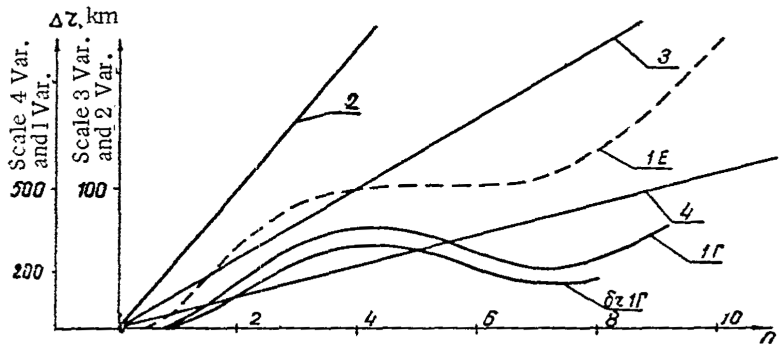


Fig. 27

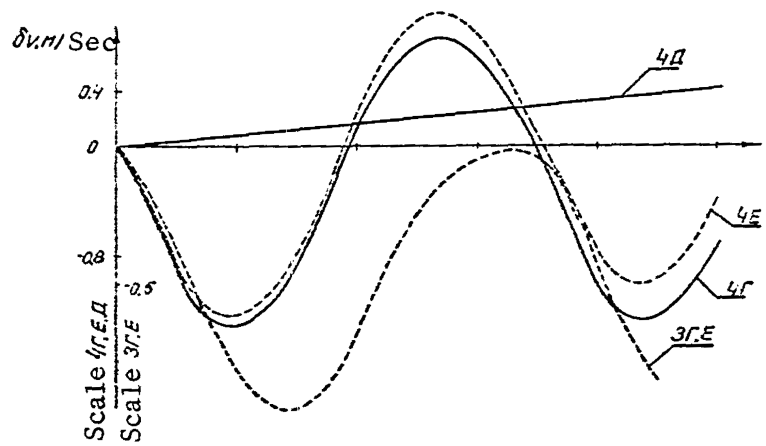


Fig. 28

This fact indicates that quasisecular disturbances of the focal parameter in the field of models Б and В is due to movement of the line of apsides, which does not take place in orbits with $i_0^* = 63.4°$. Perturbation in the field of the asymmetric Earth (model Д) is slight for polar orbits and decreases rapidly with a reduction in inclination. /82

The same may be said of eccentricity perturbation δe in the field of models Б, В and Д (Fig. 18).

Motion in the field of these models is typified by the absence of quasisecular disturbances of the longitude of the ascending node in polar orbits, the absence of quasisecular disturbances of inclination for orbits with any inclinations, and the absence of perturbations of the line of apsides for orbits with $i_0 = i_0^*$ (Figures 19, 22, 23). Perturbations $\delta\omega$ of polar orbits and $\delta\Omega$ of orbits with $i_0 = i_0^*$ show very little difference in the field of models Б, В and Д. The same type of similarity is true for disturbances δt_Ω (Fig. 24). Quasisecular disturbances of the function $\delta r(u)$ are extremely close (and low in amplitude) in the field of models Б and В (Fig. 25), noticeably greater in the field of the model which takes account of the Earth's asymmetry (model Д), and absent in the field of all three models for orbits with $i_0 = i_0^*$. This situation is also explained by the fact that the line of apsides undergoes no perturbations in such orbits.

The difference between functions $\Delta r(u)$ and $\delta r(u)$ is due chiefly to perturbations $\delta\Omega$ and δi. This also explains the characteristic form of graphs of $\Delta r(u)$ for orbits with inclinations $i_0 = 90°$ and $i_0 = i_0^*$ (Fig. 27).

The linear or nearly linear nature of disturbances of orbital elements in the analysis given above is due in a number of instances to the fact that the purely secular and long-period components (the latter being due to rotation of the line of apsides) are not separated in quasisecular disturbances, and the movements of the satellite are considered over a short time interval where changes in the disturbed functions are nearly linear.

The determination of purely secular variations in the orbital elements of artificial satellites, as has already been mentioned previously, is a complex problem which may be solved only analytically or by integration of averaged equations (see for instance the work of V. P. Taratynova, [14]). In particular, the absence of secular variations in the semimajor axis of the orbits of satellites (and planets) has been proved for motion in a field of conservative forces (to a certain approximation). Violation of this fact would lead to an unlimited increase in the energy of motion. It may be concluded from this consideration that there are no secular perturbations of /83
the focal parameter or eccentricity, which also follows from an examination of the relationship between the semimajor axis and these parameters

$$a = p/(1 - e^2)$$

assuming that a is finite.

Thus, secular disturbances affect only the angular distance of the perigee ω, the longitude of the ascending node Ω (the latter, in particular, causes additional disturbances in the field which takes account of the triaxial shape of the Earth) and the functions $t_\Omega(u)$.

It should be borne in mind that the monotonic increase in perturbations $|-\delta t_\Omega|$ is not associated with the continuous reduction in the draconic period. The function $t_\Omega(u)$ corresponds to the time of motion of the satellite from initial point $u = u_0$ to the current value of the argument. When $u = u_0 + 2n\pi$ ($n = 1, 2,\ldots,$), this corresponds to n revolutions, which may conditionally be called n draconic periods (strictly speaking, we will have draconic periods only when $u_0 = 0$, see for instance [19]). Thus, the perturbation of the draconic period δT_Ω is constant and is equal to the disturbance δt^1_Ω at the end of the first revolution. But the overall time of motion (reckoned from the initial point) at the end of each n-th revolution is equal to $t_\Omega = n(T_\Omega - \delta t^1_\Omega)$.

Disturbances of elements p, e, i, and $\delta r(u)$ in a gravitational field which takes account of the triaxiality of the geoid (models Γ and E) have a periodicity which is independent of the diurnal rotation of the Earth. The linear component is strongly pronounced in disturbances of the function $\delta\Omega(u)$ of a polar orbit in a field of models Γ and E (see Fig. 22), and when $i_0 = i_0^*$, the periodic component in the quasisecular disturbance disappears entirely. The functions $\delta\omega(u)$ and $\delta t_\Omega(u)$ also change linearly (see Figures 19 and 24).

The function $\Delta r(u)$ shows an almost strictly linear increase when $i_0 = i_0^*$ (See Fig. 27). This is explained by the linear variation in $\delta\Omega$ in this case, which affects the amplitude of $\Delta r(u)$ more strongly than do the slight oscillations in δi (see Fig. 23).

The large value of quasisecular disturbances in $\Delta r(u)$ for nonpolar orbits should be noted as well as the large quasisecular disturbances in $\delta t_\Omega(u)$, and also the insignificant difference between the disturbances of these functions in the field of various models of the gravitational field.

On these same graphs (see Fig. 17-28) are shown the quasisecular disturbances of quasisecular orbits which do not differ fundamentally from the disturbances of elliptical orbits. The main difference lies in the fact that disturbances of the focal parameter of circular orbits in the field of models Г and Е are positive rather than negative, and are low in absolute value. /84

The maximum quasisecular disturbances of elliptical orbits on an interval of ten revolutions are given in Table 8.

TABLE 8*

A	$\lvert\delta p\rvert_{max}$, m		$\lvert\delta e\rvert_{max}\cdot 10^4$		$\lvert\delta\omega\rvert_{max}$		$\lvert\delta\Omega\rvert_{max}$		$\lvert\delta i\rvert_{max}$		$\lvert\delta t\rvert^{s}_{max}$		Δr_{max}	
	1	2	1	2	1	2	1	2	1	2	1	2	1	2
Б	0,8	0	0,008	0	2°26'12"	0	0	2°12'3"	0	0	58,2	92,4	288 m	253 km
В	0,65	0	0,007	0	2°25'41"	19"	0	2°11'42"	0	0	58,5	92,5	280 m	252 km
Г	219	274	0,75	0,725	2°25'50"	29"	2",47	2°12'24"	44"	21"	58,2	94,7	500 m	254 km
Д	53	0	0,79	0	2°26'18"	9"	0	2°12'4"	0	0	58,1	92,4	840 m	253 km
Е	198	274	1,05	0,725	2°25'28"	1"	2",47	2°12'3"	44"	21"	58,5	94,8	1032 m	253 km

Key A = Model of Gravitational Field

Note: Given in column 1 are the disturbances for an orbit with the following parameters: $\Omega_0 = 0$; $i_0 = \pi/2$; $\omega_0 = 0$; $e_0 = 0.0499$; $p_0 = 6{,}996$ km (the altitude of the apogee $h_A = 1{,}000$ km, the altitude of the perigee $h_\Pi = 300$ km). Given in column 2 are the disturbances for an orbit with the following parameters: $\Omega_0 = 0$; $i_0 = 63°26'$; $\omega_0 = 0$; $e_0 = 0.0499$; $p_0 = 6{,}996$ km.

Disturbances from Gravitational Anomalies

Since the term of the potential expansion containing harmonic P_{20} has the principal disturbing effect, the field of a spheroid may be taken as the normal field. In this case, the square of flattening, asymmetry of the hemispheres and triaxiality of the Earth should be included with gravitational anomalies (see Chapter One).

The periodic and quasisecular disturbances of orbital elements due to gravitational anomalies are given in Figures 29-44. The initial parameters of the orbits are given in Table 5.

The following notation is used on these graphs for disturbances from anomalies: И-- disturbances from the square of flattening; Н -- disturbances

*Tr. Note: Commas indicate decimal points.

from asymmetry; M-- disturbances from triaxiality; H-- disturbances from the sum total of the enumerated anomalies.

Periodic disturbances of elliptical orbits due to the second power of flattening and asymmetry of the hemispheres are characterized by the appearance of upper harmonics. The effect of asymmetry on disturbances of the focal parameter, eccentricity and line of apsides decreases with a reduction in the inclination. The nature of the disturbances in this case becomes more harmonic (Fig. 29-33).

Disturbances in the longitude of the ascending node from the second power of flattening and asymmetry are comparatively high. These disturbances amount to several angular seconds (Fig. 34).

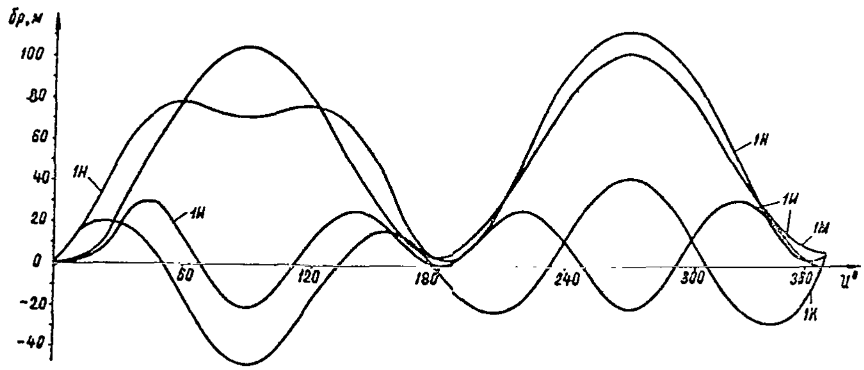


Fig. 29

In general, the effect of these two anomalies on periodic disturbances /100
of all parameters is identical and small. In particular, this effect does not exceed the following values: $|\delta p| \leq 50$ m; $|\delta e| \leq 0.5 \cdot 10^{-5}$; $\Delta r \leq 70$ m; $|\delta t| \leq 0^s.01$. The effect of triaxiality is approximately an order of magnitude greater and leads to the appearance of harmonics similar to the fundamental harmonics (caused by the second zonal harmonic). All enumerated anomalies lead to periodic disturbances of functions Δr and $|\delta t|$ not exceeding 500 m and $0^s.45$, respectively (Figures 36, 38).

The maximum quasisecular disturbances resulting from triaxiality do not exceed: $\delta p \leq 280$ m; $\delta e \leq 0.075 \cdot 10^{-3}$; $|\delta\omega| \leq 25''$; $|\delta\Omega| \leq 20''$; $|\delta i| \leq 22''$; $/\delta t/ \leq 2^s.5$; $\Delta r \leq 280$ m.

The square of flattening chiefly affects the quasisecular motions of the line of apsides and the line of nodes. They are comparable to disturbances from triaxiality. Asymmetry of the hemispheres has the greatest effect on perturbation of eccentricity (in this case reaching a value of the order of 10^{-7}) and the function Δr. /100

On the tenth revolution, the quasisecular perturbation in Δr due to asymmetry is equal to 500 meters, and is greater than the disturbance due to triaxiality. Thus, if the effect of triaxiality is taken into account, and also the asymmetry of the Earth for orbits with $i_0 \neq i_0^*$, then the gravitational anomalies which are disregarded over an interval of about one day will lead to an error of $\Delta r \leq 100$ meters in the position of the satellite.

It should be noted that neither the square of polar flattening nor asymmetry cause any quasisecular perturbations of the focal parameter or inclination of orbits with inclination $i_0 = i_0^* = 63.4°$, that the square of flattening for orbits with this inclination causes no quasisecular perturbations of eccentricity or the function $\delta r(u)$, and that asymmetry causes no perturbations in the ascending node.

Triaxiality leads to disturbances with a diurnal period in functions δp, δe, $\delta \Omega$, δi and δr. In orbits with inclination $i_0 \neq i_0^*$, perturbations of this type are also typical for the function $\delta\omega(u)$ (Figures 39-42, 44).

The asymmetry and triaxiality of the Earth cause a strong change in the periodic perturbations of the line of apsides at the end of the draconic period in circular orbits. The quantity $\delta\omega$ may reach 80° under the effect of each of the three anomalies. For circular orbits with an inclination close to 63.4°, triaxiality produces a discontinuity in the function $\delta\omega$ and the points u = 140° and 220°. This singularity disappears with a reduction in the amount of inclination. However, for nearly equatorial orbits there is a /101 characteristic increase in disturbances of the function $\delta\omega(u)$ at the end of the period, which is caused by triaxiality and asymmetry. For circular orbits, large perturbations in the function δr appear at the end of the period ($|\delta r| \approx 1{,}000$ m), which is explained by the high amplitude of the long-period oscillation (due to rotation of the Earth).

Given in Table 9 are periodic disturbances as a function of the various gravitational anomalies.

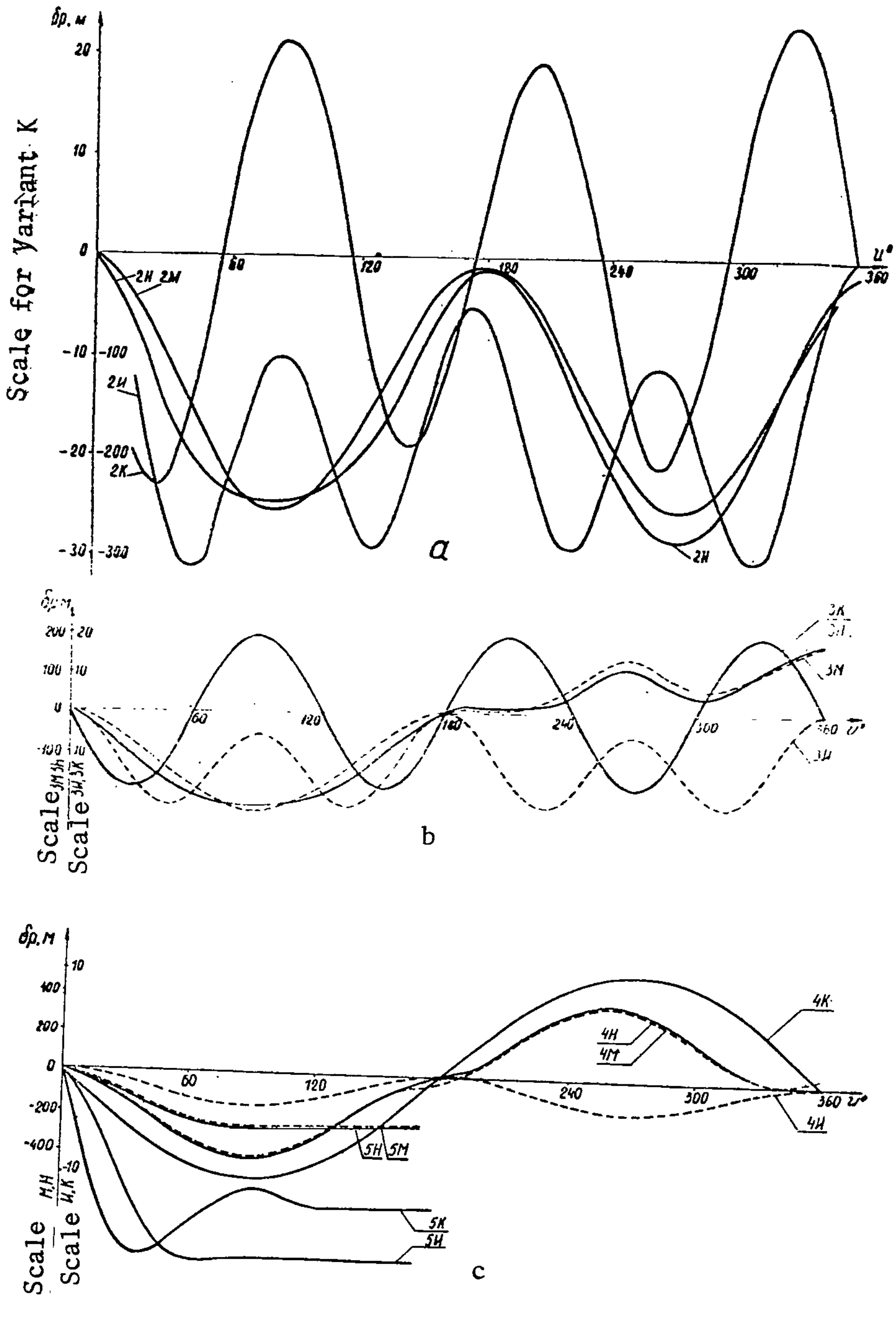
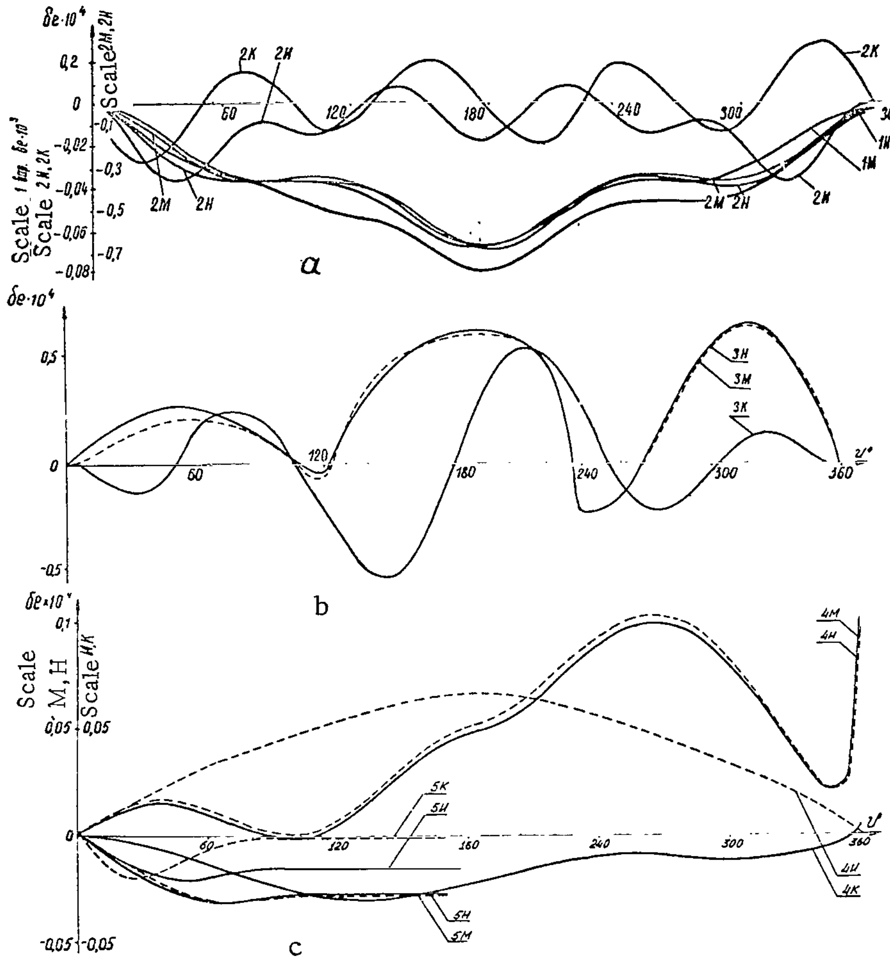
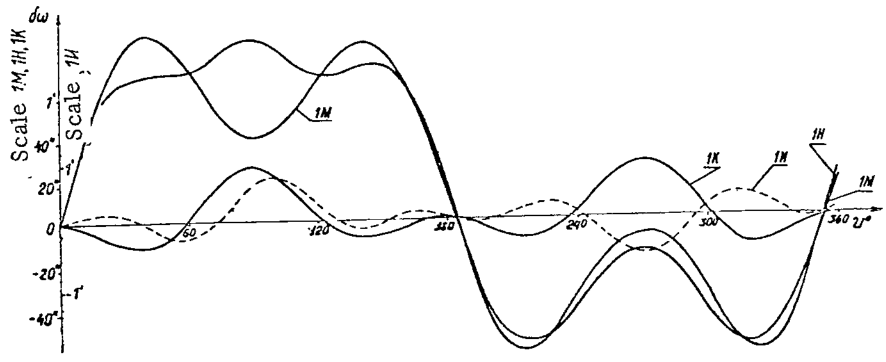


Fig. 30



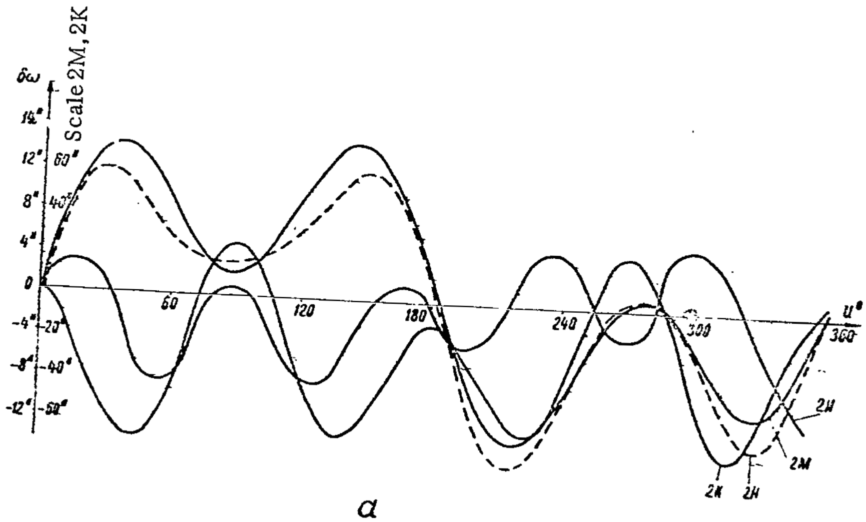
Scale 1 kp. δe·10³
Scale 2M, 2K
Scale 2M, 2H
2K
2H
2M
1H
1M
a
b
3H
3M
3K
Scale M, H
Scale HK
4M
4H
4K
5K
5H
5M
c

Fig. 31

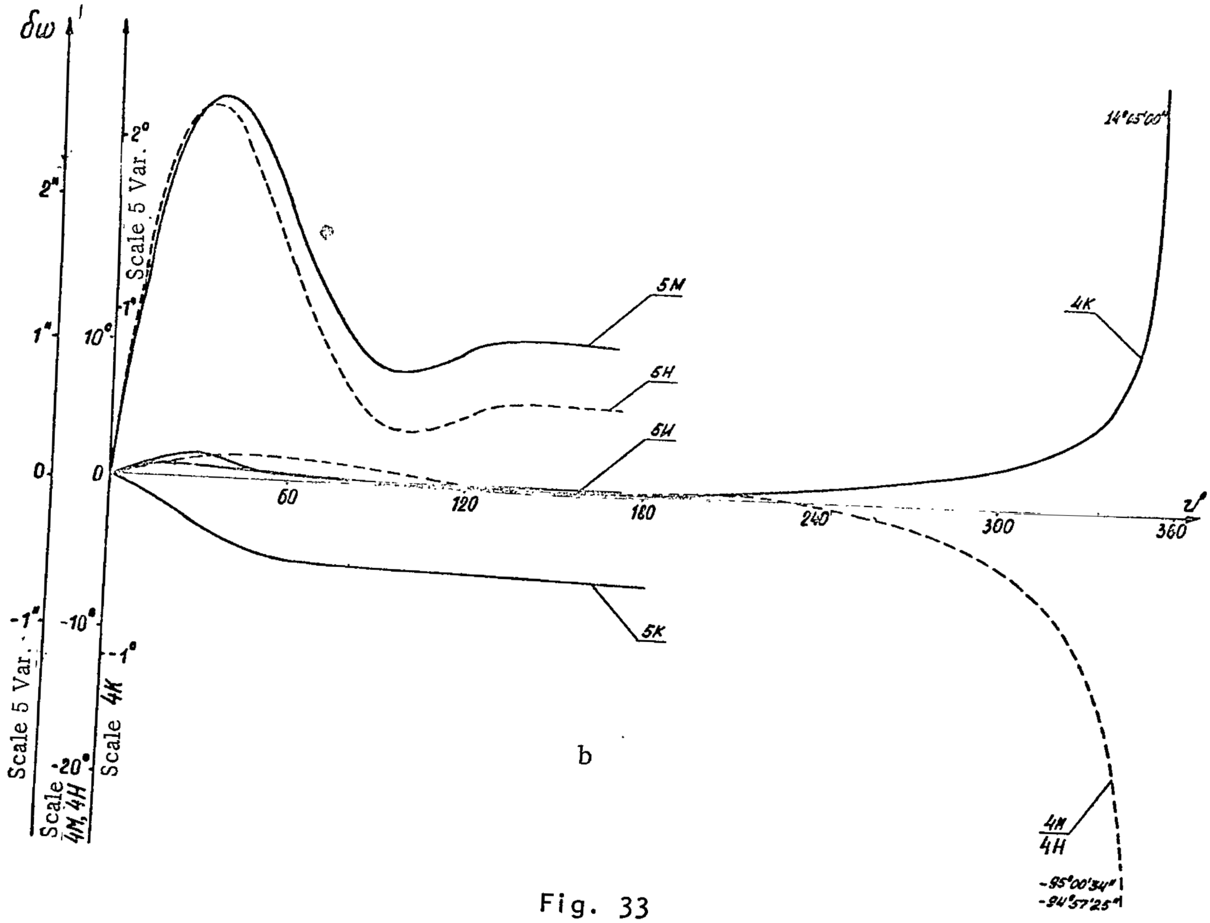


Scale 1M, 1H, 1K
Scale 1H
1M
1K
1H

Fig. 32

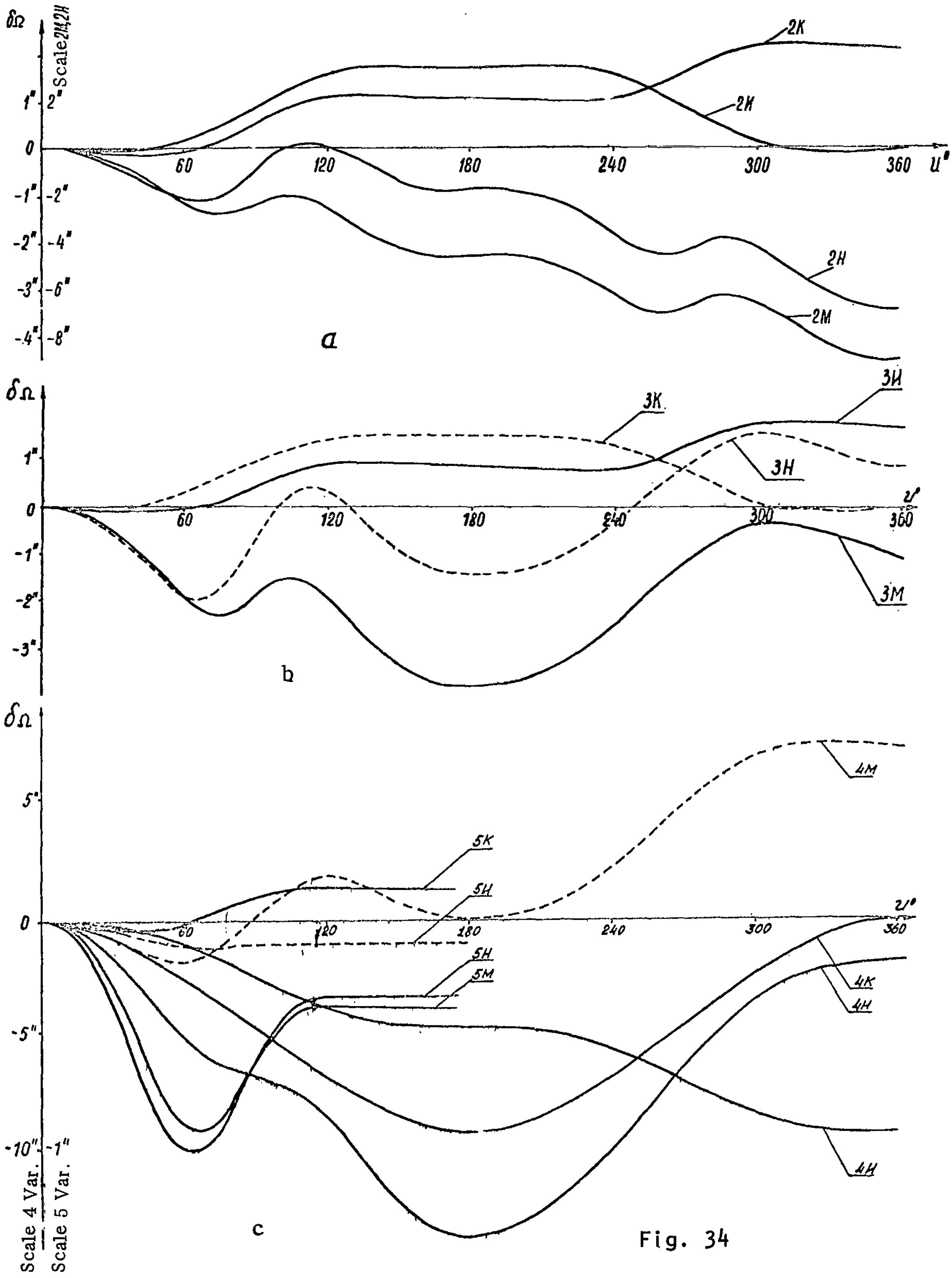

δω
Scale 2M, 2K
14"
12"
60"
8"
40"
4"
0
-4"
-20"
-8"
-40"
-12"
-60"
60
120
180
240
300
360
u°
2H
2M
2K
2H

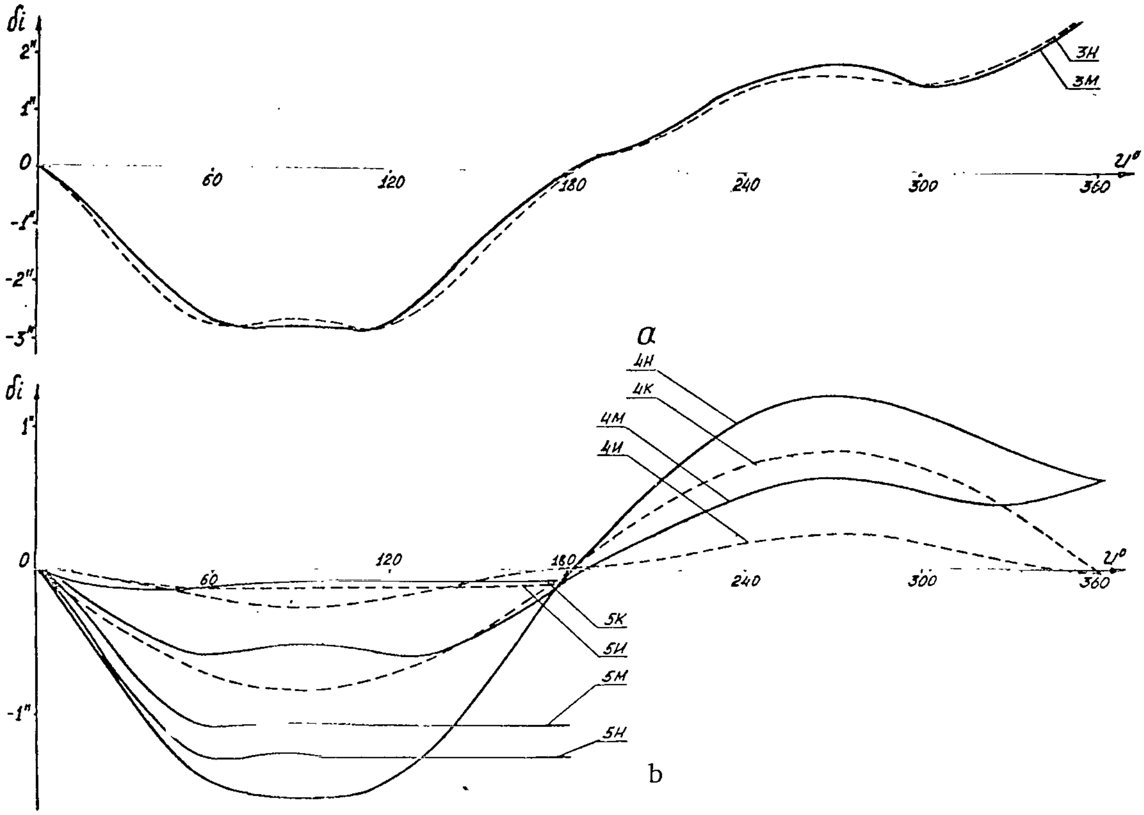
a


δω
2"
1"
10°
0
-1"
-10°
-20°
2°
1°
-1°
Scale 5 Var.
Scale 5 Var.
Scale 4M, 4H
Scale 4K
5M
5H
6H
5K
4K
4M
4H
60
120
180
240
300
360
14°45'00"
-95°00'34"
-94°57'25"

b

Fig. 33


δΩ
Scale 2M,2H
1"
2"
0
-1"
-2"
-2"
-4"
-3"
-6"
-4"
-8"
60
120
180
240
300
360
u°
2K
2K
2H
2M
a
δΩ
1"
-1"
-2"
-3"
3K
3H
3H
3M
b
δΩ
5"
-5"
-10"
-1"
5K
5H
5H
5M
4M
4K
4H
4H
Scale 4 Var.
Scale 5 Var.
c
Fig. 34



δi
2"
1"
0
-1"
-2"
-3"
60
120
180
240
300
360
ν°
3Н
3М
a
4Н
4К
4М
4И
1"
-1"
5К
5И
5М
5Н
b

Fig. 35

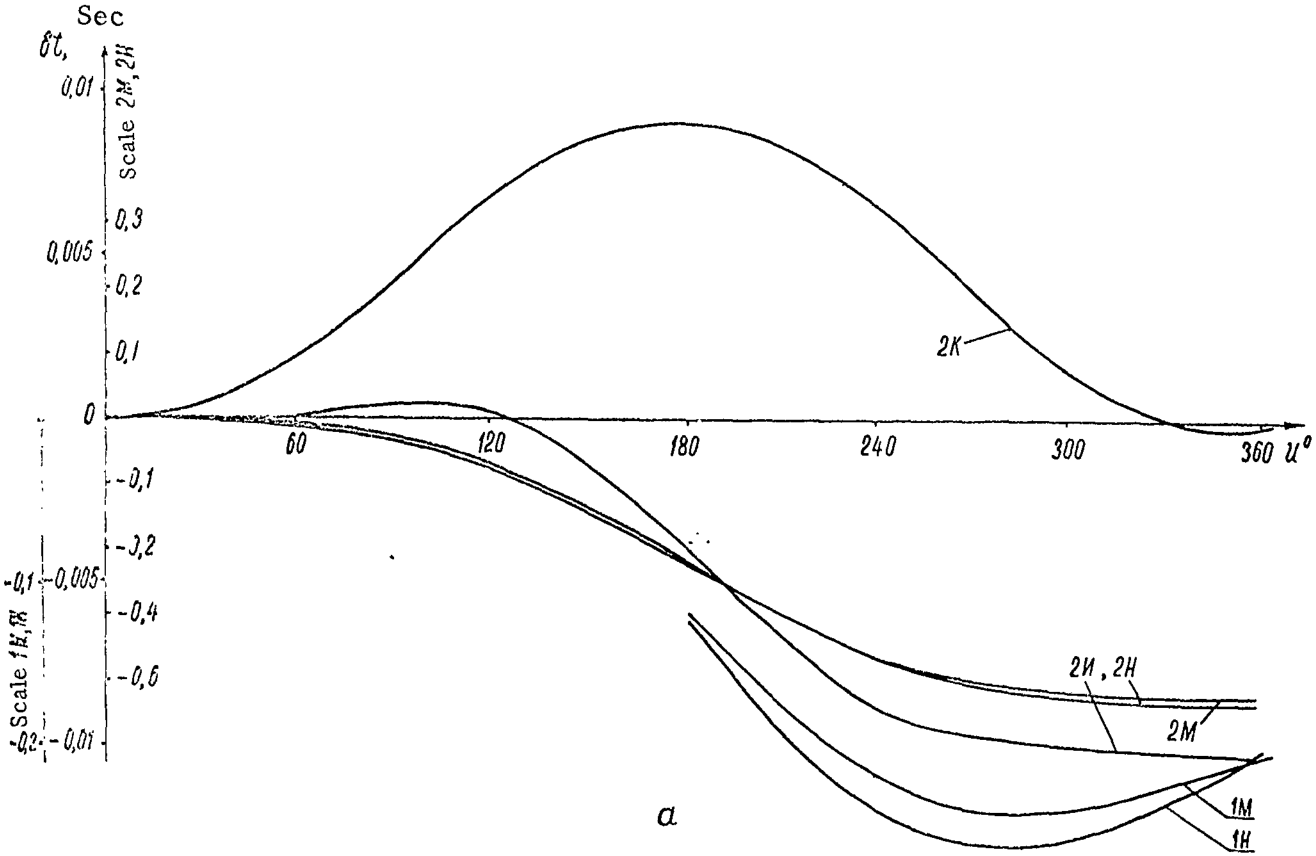

Sec
δt,
0,01
0,005
0
-0,005
-0,01
Scale 2M, 2H
0,3
0,2
0,1
-0,1
-0,2
-0,4
-0,6
Scale 1H, 1M
-0,1
-0,2
60
120
180
240
300
360 u°
2K
2И, 2H
2M
1M
1H
a

Fig. 36 a

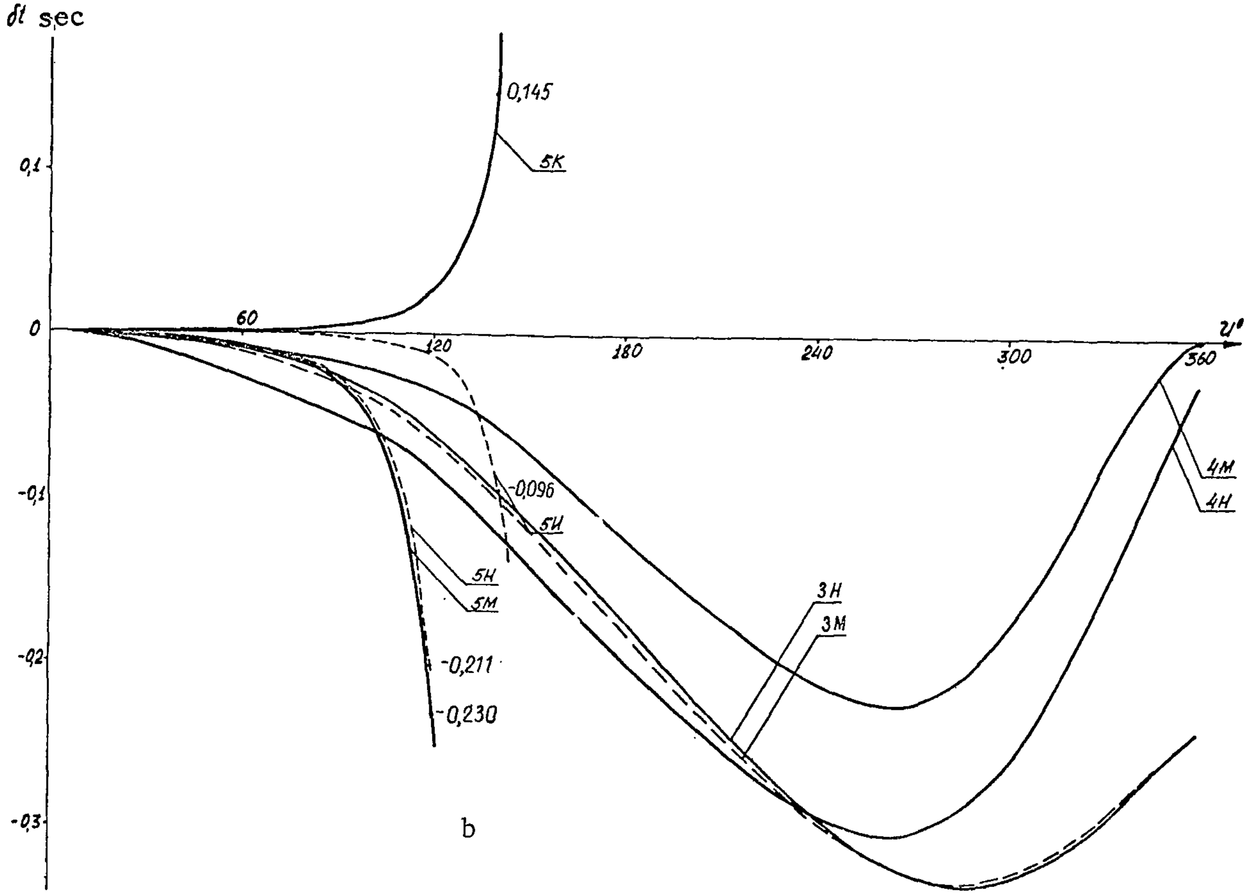

δt sec
0,1
0
-0,1
-0,2
-0,3
60
120
180
240
300
360
υ°
0,145
5K
-0,096
5И
5H
5M
-0,211
-0,230
3H
3M
4M
4H
b

Fig. 36 b

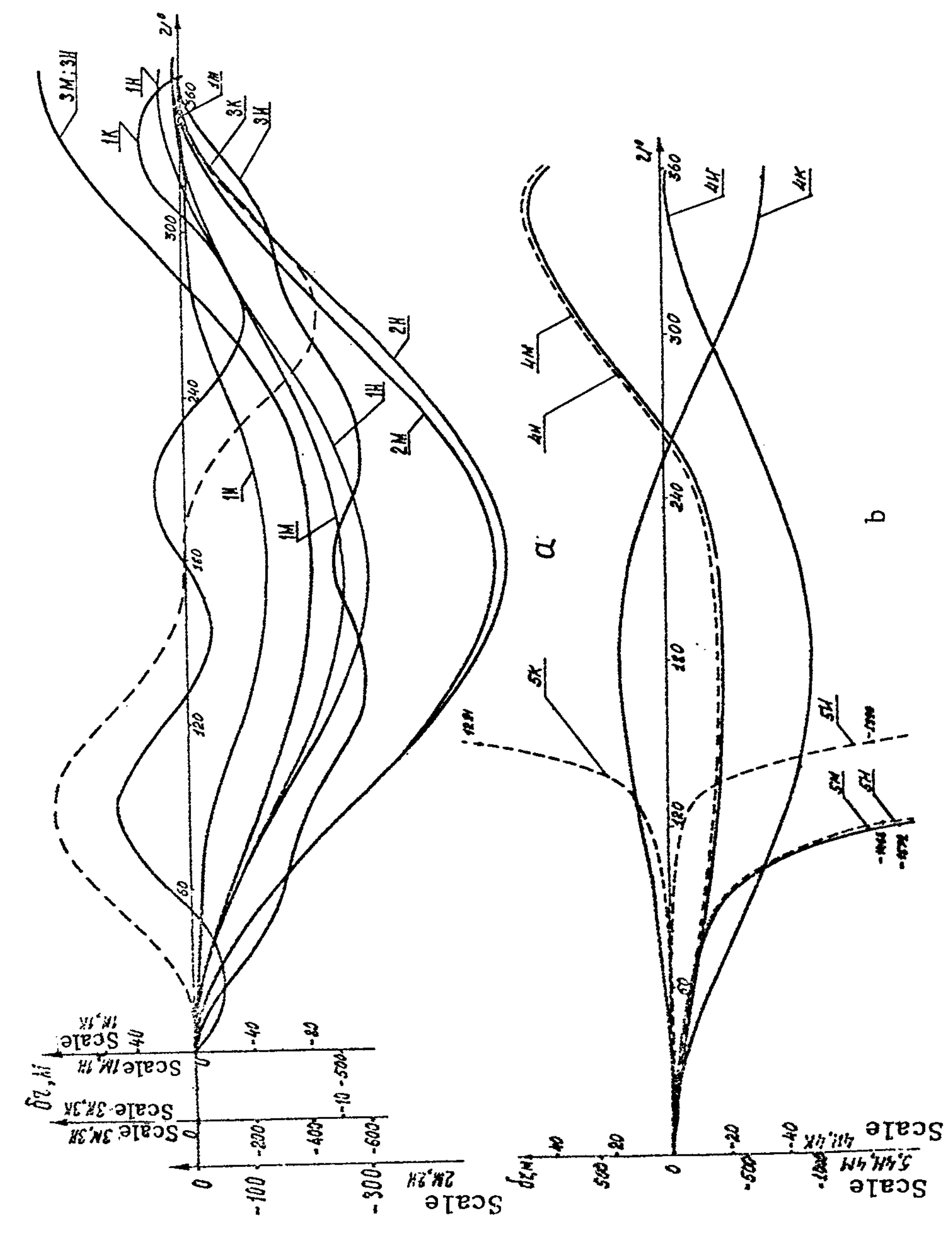
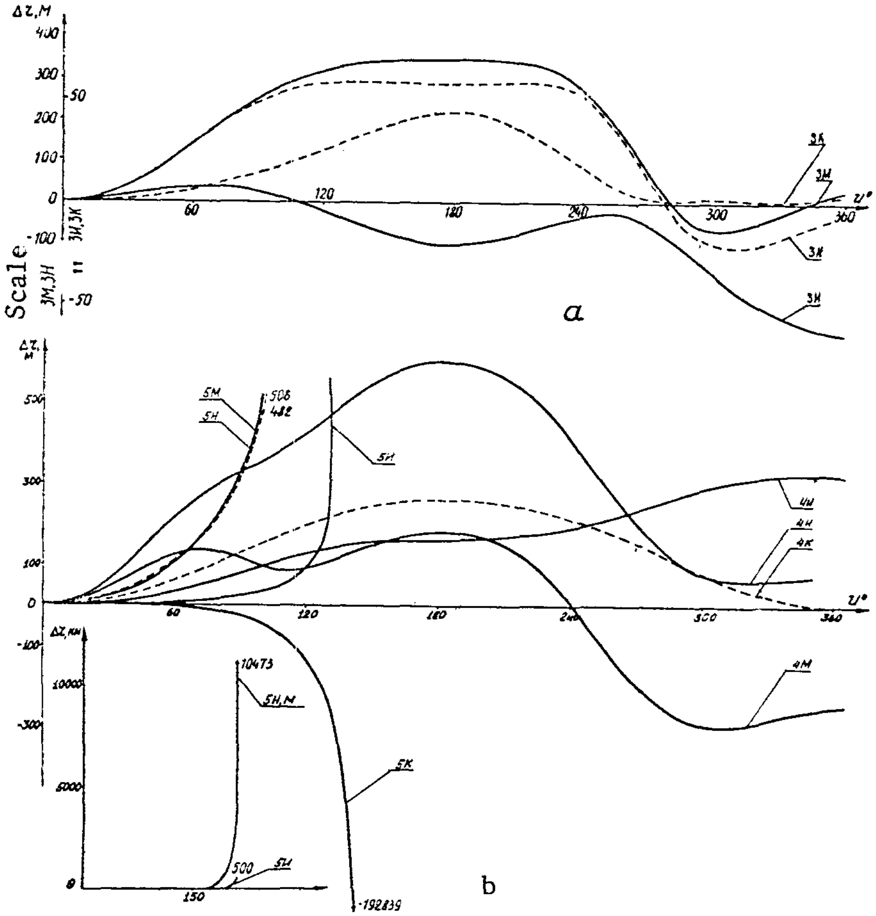
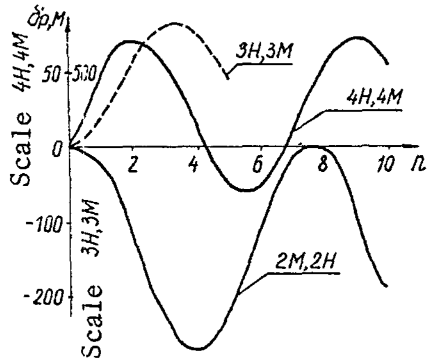

Scale
Scale 1M, 1H
Scale 3M, 3H
Scale 3K, 3K
2M, 2H
1M, 1K
3M; 3H
1K
1H
3K
3H
2H
2M
1M
4M
4H
4K
5K
5H
a
b
0
60
120
180
240
300
360
-100
-300
-200
-400
-600
40
-40
-80
20
-20
500
-500

Fig. 37



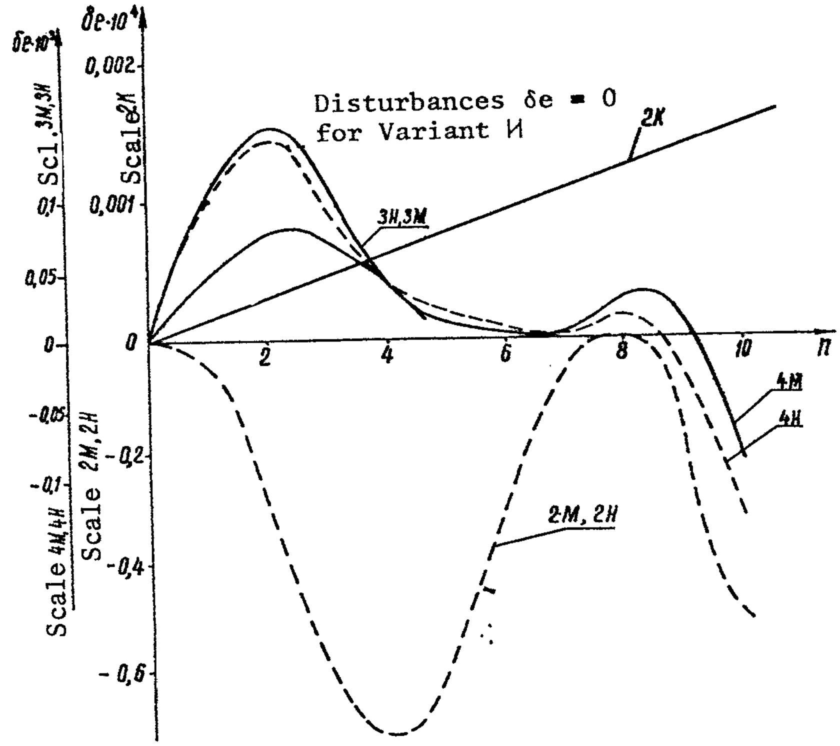
a
b
Scale
3M,3H
3H,3K
3K
3M
3H
5M
5H
4M
4H
4K
5H,M
5K
10473
-192839

Fig. 38



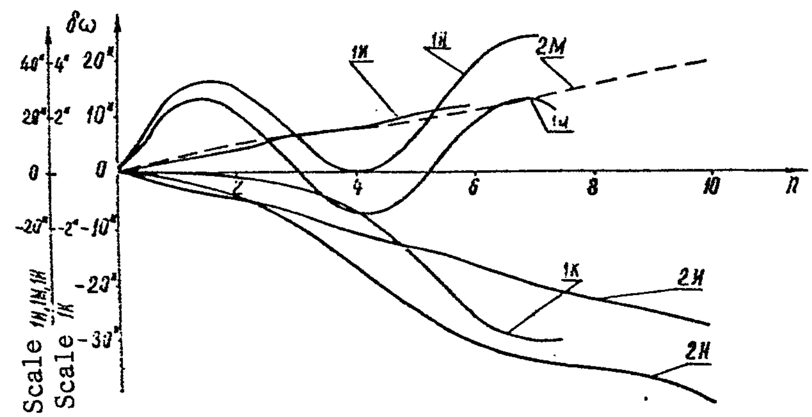
Scale 4H,4M
Scale 3H,3M
3H,3M
4H,4M
2M,2H

Fig. 39



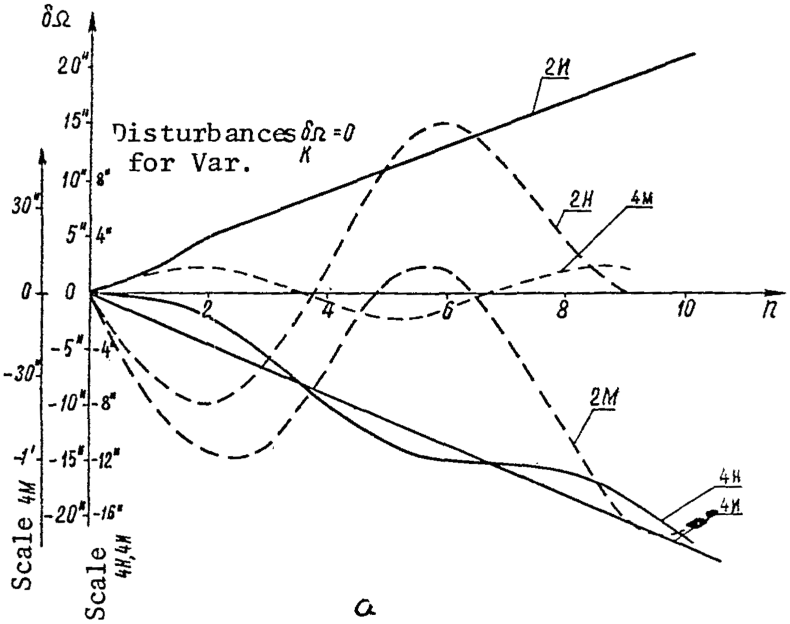
$\delta e \cdot 10^4$
0,002
0,001
0,1
0,05
0
-0,05
-0,1
-0,2
-0,4
-0,6
Scl. 3M,3H
Scale 2K
Scale 4M,4H
Scale 2M,2H
Disturbances δe = 0
for Variant И
2K
3H,3M
4M
4H
2M, 2H
2
4
6
8
10
n

Fig. 40

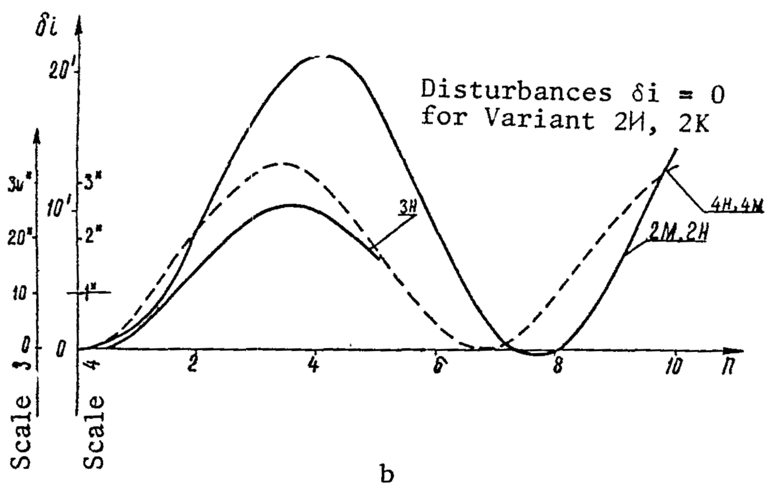


δω
40"
4"
20"
20"
2"
10"
0
-20"
-2"
-10"
-20"
-30"
Scale 1M,1H,1H
Scale 1K
1H
2M
1K
2H
2H
2
4
6
8
10
n

Fig. 41


δΩ
20"
15"
Disturbances δΩ = 0
K
for Var.
10" 8"
30"
5" 4"
0 0
-5" -4"
-30"
-10" -8"
-1' -15" -12"
-20" -16"
Scale 4M
Scale 4H,4M
2H
2H
4M
2M
4H
4M
2
4
6
8
10
n

a


δi
20'
Disturbances δi = 0
for Variant 2H, 2K
30"
10' 3"
20" 2"
3H
4H,4M
2M,2H
10 1"
0 0
Scale 3
Scale 4
2
4
6
8
10
n

b

Fig. 42

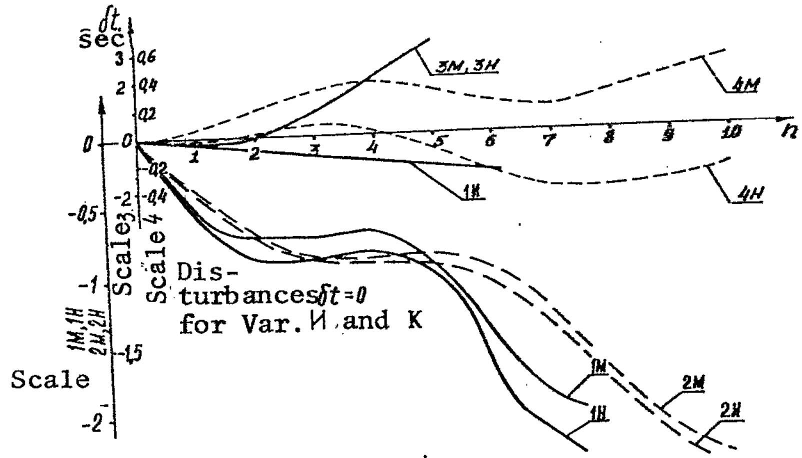


Fig. 43

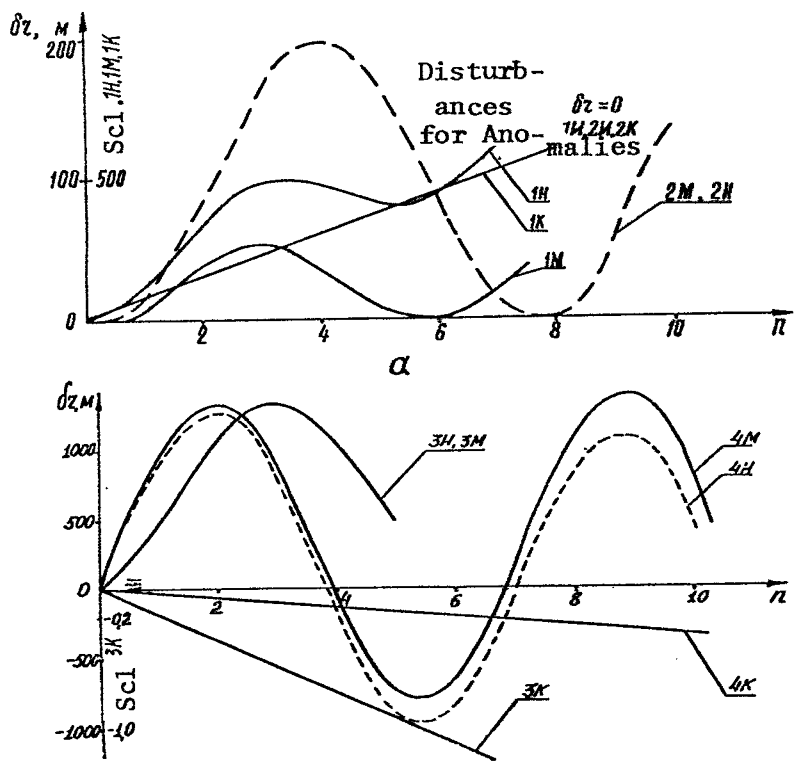
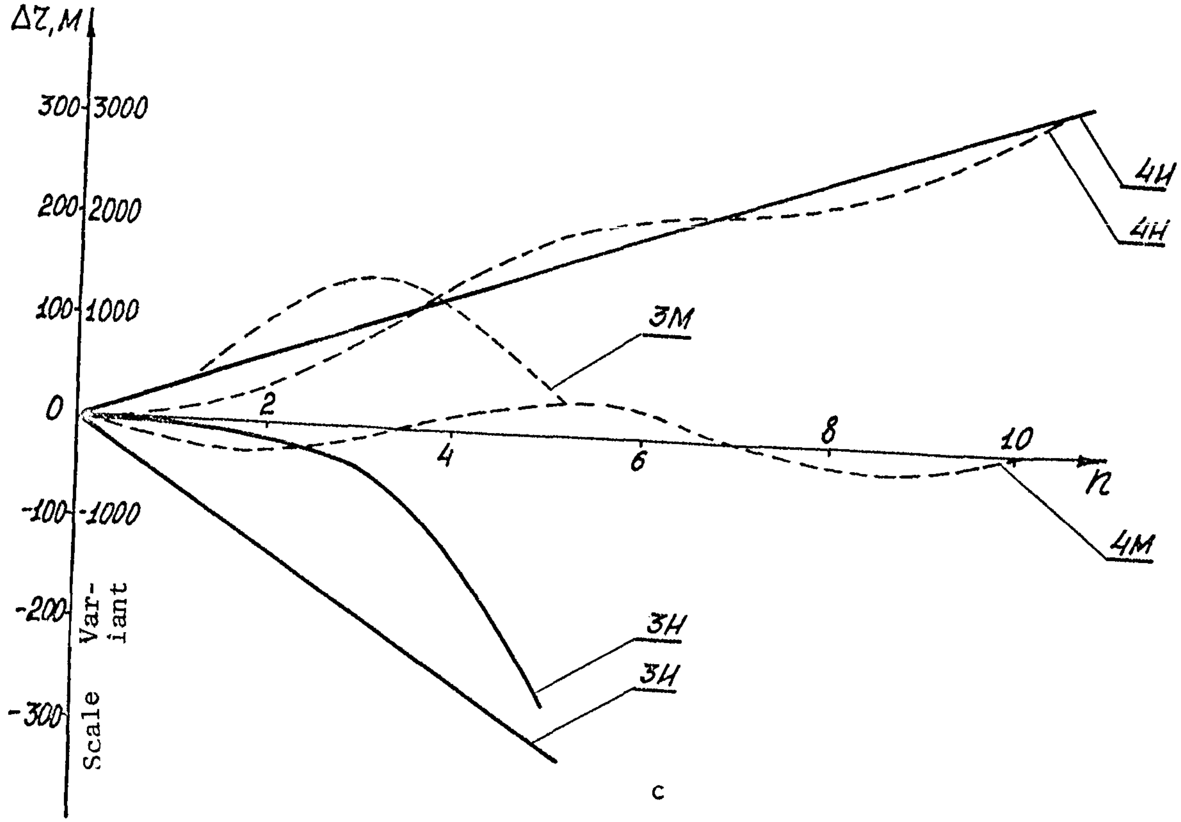


Fig. 44 a & b



ΔZ,M
300 3000
200 2000
100 1000
0
-100 -1000
-200
-300
Var-
iant
Scale
2
4
6
8
10
n
4H
4H
3M
4M
3H
3H
c

Fig. 44 c

TABLE 9*

	Circular orbits $p_0 = 7400$ km $i_0 = 63°,4$		Hyperbolic orbits $e_0 = 1,01$; $p_0 = 14800$ km $u = 170°$	
	Δr_{max}, м	$\lvert\delta t\rvert^{S}_{max}$	Δr_{max}, km	$\lvert\delta t\rvert^{S}_{max}$
Effect of the square of flattening	65 (323)	0,0040 (0,04)	500	470
Effect of asymmetry of the hemispheres	44 (263)	0,0078 (0,04)	200	220
Effect of triaxiality of the earth	354 (294)	0,3307 (0,221)	10500	
Overall effect of the anomalies	300 (603)	0,3316 (0,301)	10100	6170

Note: Shown in parentheses are the figures for a circular orbit with $i_0 = 10°$.

§8. Effect of Orbital Parameters on Disturbances in Satellite Motion in the Field of a Spheroid

The effect which initial orbital parameters have on perturbations is very important in planning the orbits of artificial Earth satellites [4]. It was shown in the preceding section that the field of a spheroid is the strongest factor which perturbs satellite motion. It is natural therefore, in the given case to give separate consideration to the effect which orbital parameters have on disturbances in the field of a spheroid (treating it as a normal gravitational field) and on disturbances from gravitational anomalies [51].

Since the field of a triaxial ellipsoid has a characteristic effect on quasisecular perturbations of a number of elements, individual attention is also given to the effect of initial parameters on motion in this field.

Effect of Orbital Inclinations. The effect of orbital inclinations is considered over the range of angles $0 \leq i_0 \leq \pi/2$ and is shown in Figures 45-60 (for satellite motion in a field of model Б). The corresponding variants of initial conditions are given in Table 10. /107

*Tr. Note: Commas indicate decimal points.

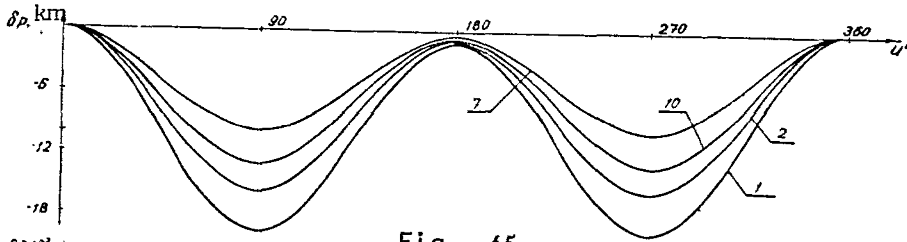


Fig. 45

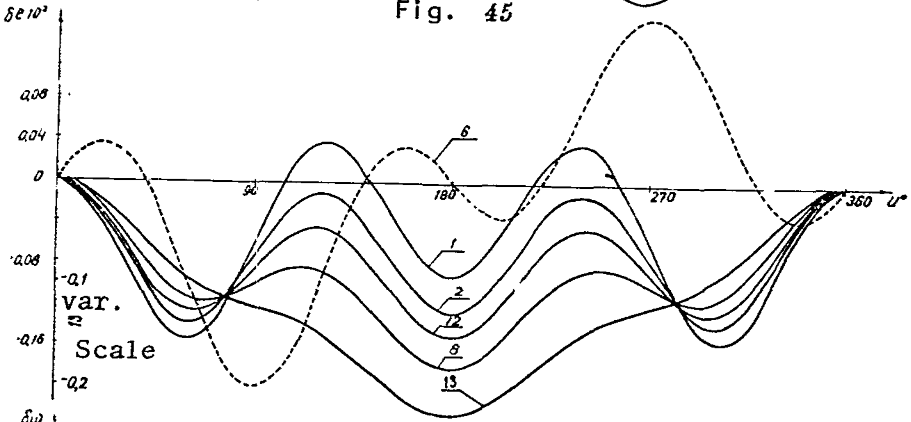


Fig. 46

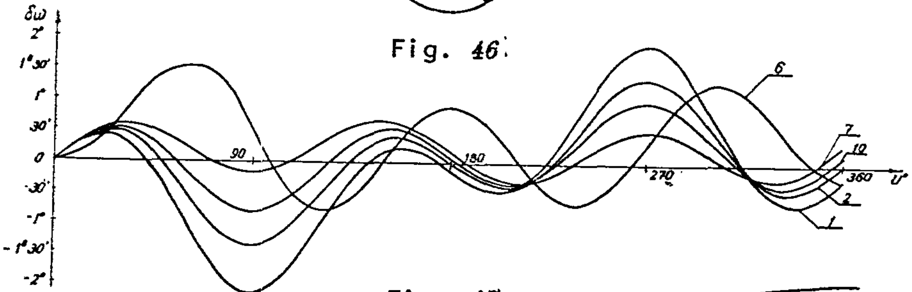


Fig. 47

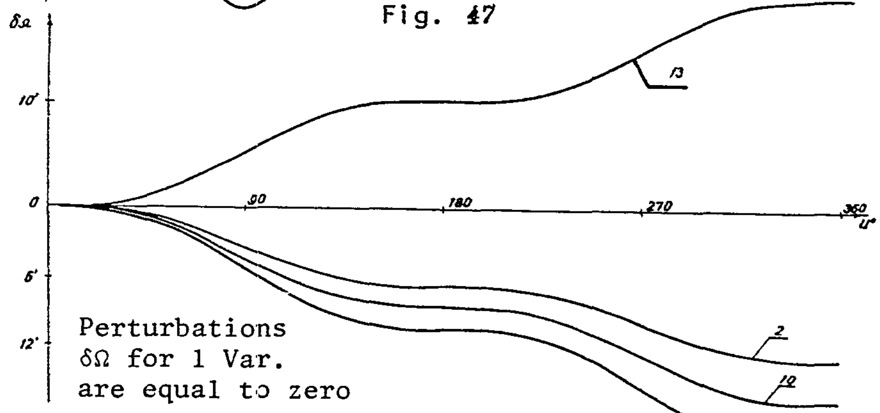


Fig. 48

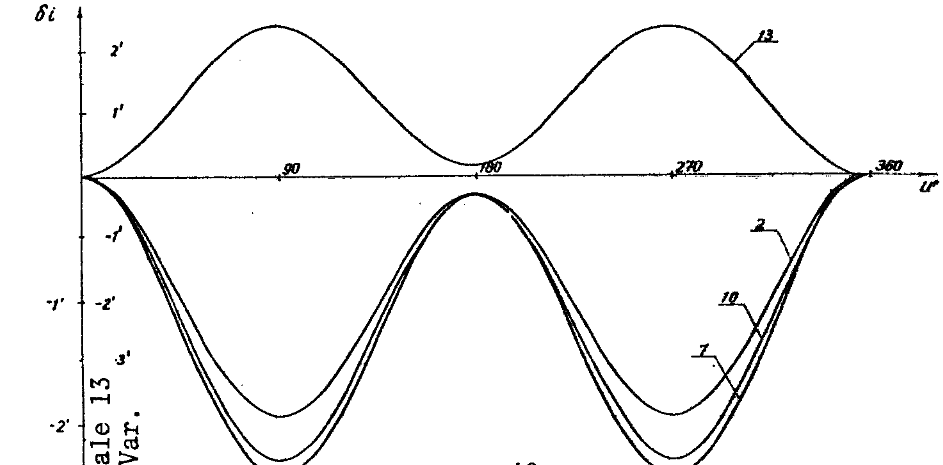


Fig. 49

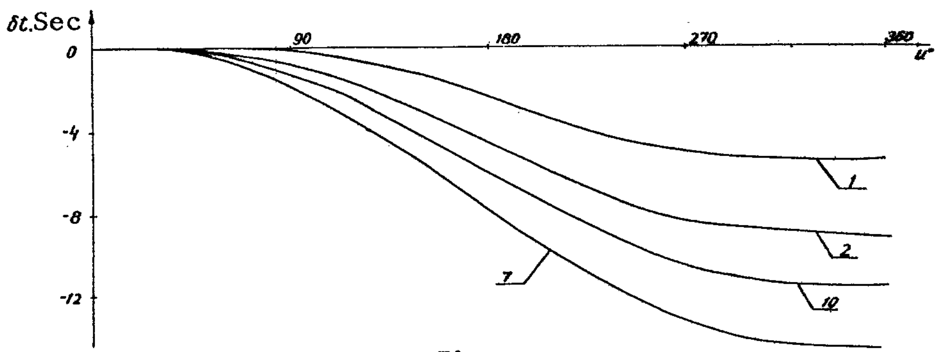


Fig. 50

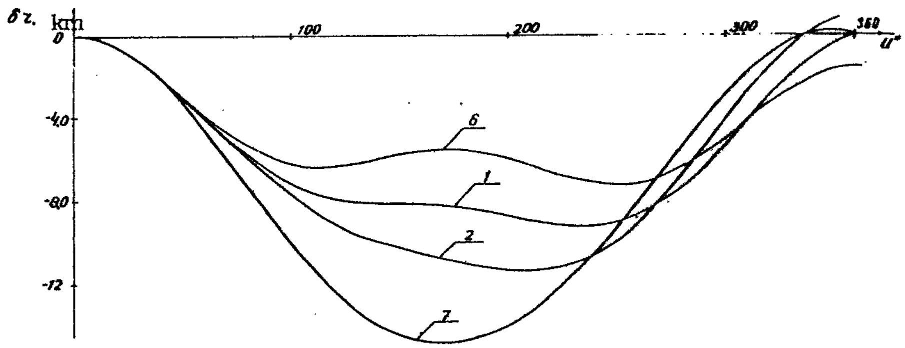
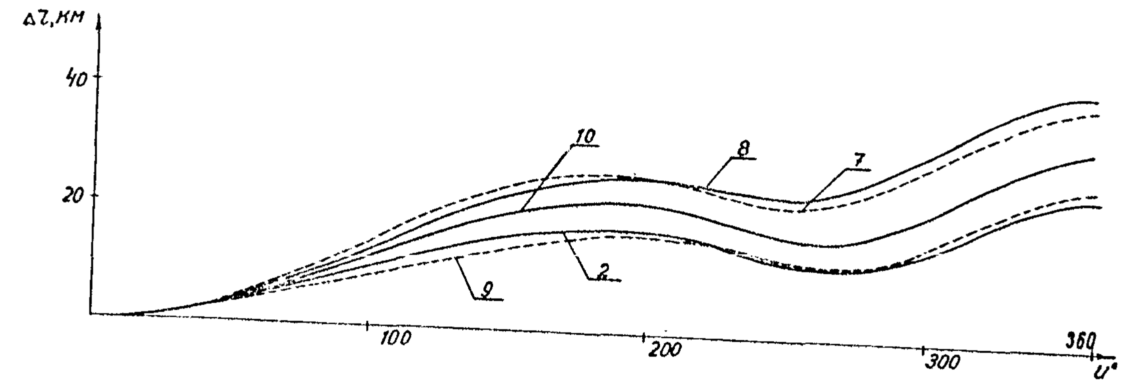
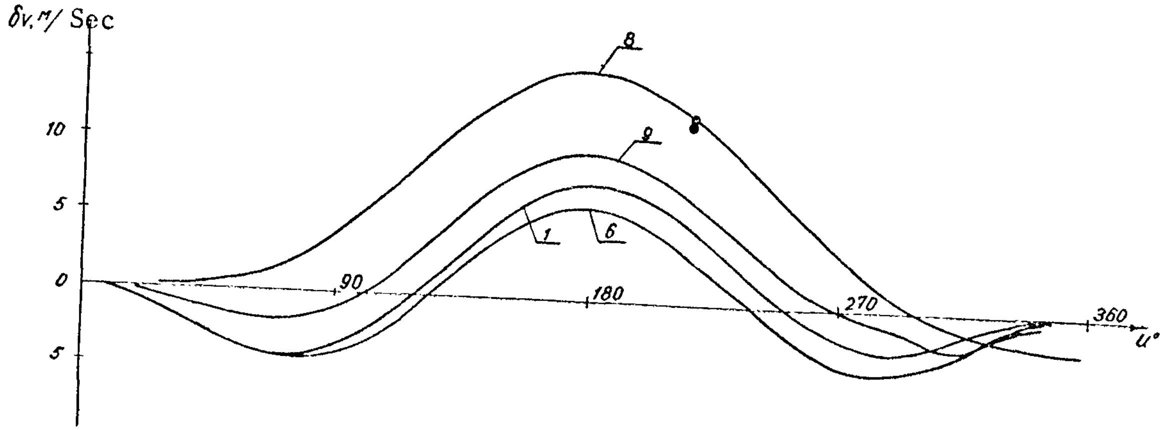


Fig. 51



Δl, км
40
20
100
200
300
360
u°
10
8
7
2
9

Fig. 52



δv, м/ Sec
10
5
0
5
90
180
270
360
u°
8
9
1
6

Fig. 53

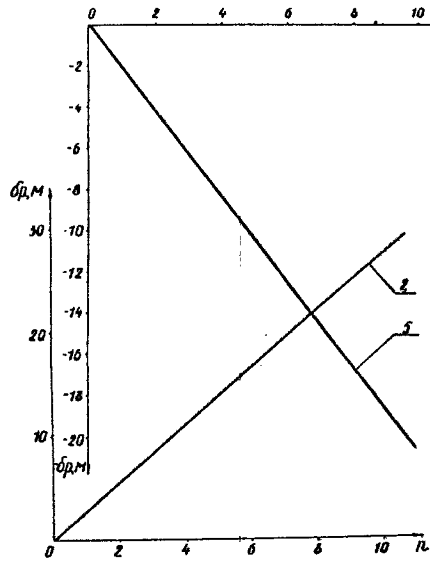


Fig. 54

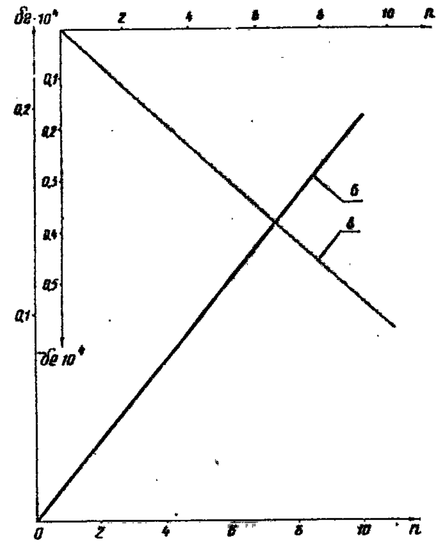


Fig. 55

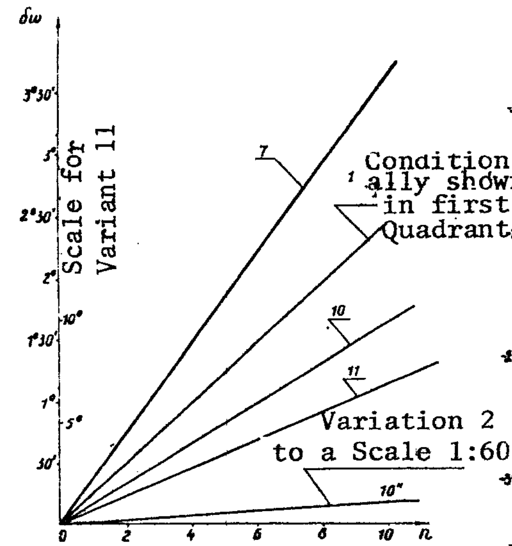


Fig. 56

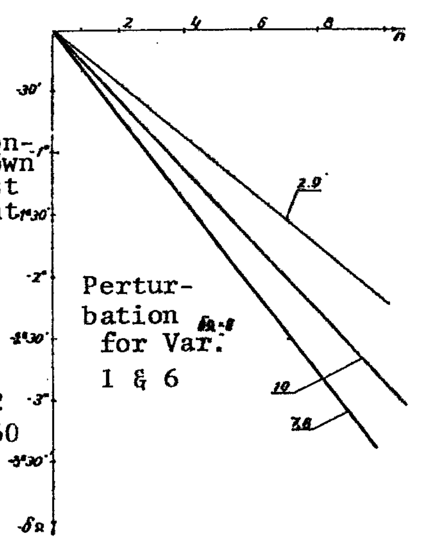


Fig. 57

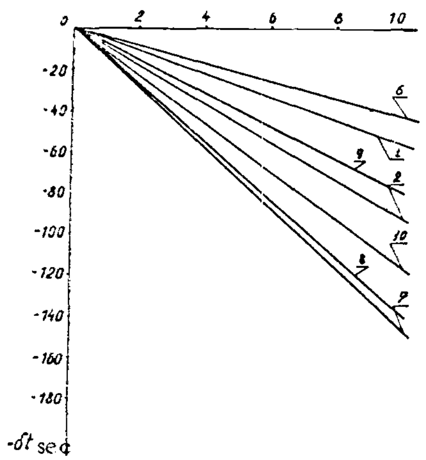


Fig. 58

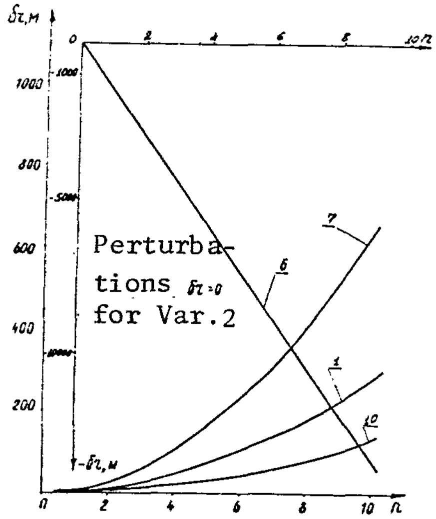


Fig. 59

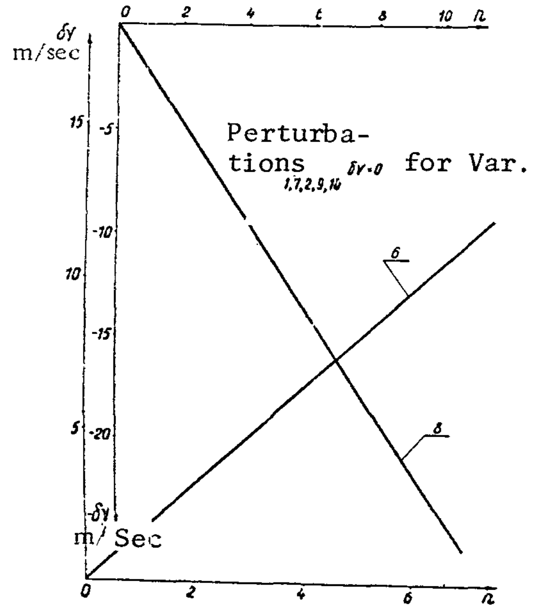


Fig. 60

TABLE 10 *

Number of Variant	Ω_0	i_0	p_0, км	e_0	ω_0	Altitude Apogee h_A, км	Altitude Perigee h_Π, км
6	0	90°	6996,087	0,0499034	90°	1000	300
7	0	45°	6996,087	0,0499034	0	1000	300
8	0	45°	6996,087	0,0499034	90°	1000	300
9	0	63°26′	6996,087	0,0499034	90°	1000	300
10	0	54°44′	6996,087	0,0499034	0	1000	300
11	0	10°	6996,087	0,0499034	0	1000	300
12	0	30°	6996,087	0,0499034	0	1000	300
13	0	135°	6996,087	0,0499034	0	1000	300

A reduction in inclination has the strongest effect on the nature of perturbations of the eccentricity and line of apsides by smoothing out their oscillations (Fig. 46, 47). Rotation of the line of apsides of circular polar orbits is replaced by oscillatory motion with a reduction in i (See Appendix VI). /107

Perturbation of the angular distance of the perigee $\delta\omega$ in elliptical nearly equatorial orbits is generally sinusoidal with period 2π and a certain secular variation, while $\delta\omega(u)$ for circular orbits changes according to a linear law (see Fig. 6).

Circular orbits show a minimum in δe_{max} when $i_0 = 63.4°$. This may be illustrated by Fig. 5 and Table 11.

TABLE 11*

	Inclination i_0			
	90°	63°,4	45°	0°
δe_{max} when $e_0 = 0$	0,001738	0,001304	0,001645	0,002470

Fig. 45 shows that a reduction in inclination considerably reduces perturbations of the focal parameter, which are generally equal to zero for equatorial orbits.

* Tr. Note: Commas indicate decimal points.

A reduction in the initial inclination of the orbit has a reverse effect on perturbations of the line of nodes and inclination (Figures 48, 49).

With a reduction in inclination, the function $\delta r(u)$ takes on the nature of a negative half-wave (Fig. 51), while Δr_{max}, as may be seen from Fig. 52, increases considerably (at inclinations of 90°, 63.4° and 45°, the quantity Δr_{max} is equal to 9.27, 25.29 and 40.13 km, respectively). /108

An increase in inclination, as may be seen from Fig. 50, results in a considerable increase (in absolute value) in the time of satellite motion δt, although the nature of this function does not change.

The effect of inclination on periodic perturbations of circular orbits is shown in Figures 5-16 and in Table 12.

TABLE 12 *

		Inclination i_0			
		90°	63°,4	45°	0°
$e_0 = 0$	δp_{min}, km	−18,12	−14,05	− 9,06	0
$p_0 = 7378$ km	δt^s	− 3,85	− 6,19	− 9,68	−15,52
	$\lvert\delta V\rvert_{max}$, sec	6,03	8,45	12,10	–
$e_0 = 0{,}0499$, $p_0 = 6998$ km	δt^s	− 5,60	− 9,20	−14,70	−15,75

When the inclination crosses over into the second quadrant, all parameters of motion except Ω and i vary in the same way as when $0 \leq i \leq \pi/2$. The functions $\delta\Omega(u)$ and $\delta i(u)$ in this quarter change sign, i.e. reverse motion of the node is replaced by forward motion, and perturbations δi are always directed toward a reduction in the angle between the plane of the orbit and the equator (see Figures 48, 49).

Quasisecular perturbations of the focal parameter, eccentricity (Figures 54, 55) and line of apsides (Fig. 56) pass through zero at $i_0^* = 63.4°$. The perturbations of the functions δr, δq and δk have a minimum at this inclination (Fig. 59), while disturbances of the orbital velocity δV increase with a reduction in inclination (Fig. 60). Perturbations δp and δe are due to long-period oscillations caused by motion of the line of apsides. As may be seen from the graphs, disturbances $\delta\omega$ are not equal to zero at the point of the minimum. This is due to the long-period components in the quantity ω. The inclination of the orbit does not undergo any quasisecular disturbances in the field of a spheroid.

Perturbations $\delta\Omega$(Fig. 57), δt (Fig. 58) and Δr, just as perturbations

* Tr. Note: Commas indicate decimal points.

δV, increase in absolute value with a reduction in inclination[1]. When the inclination enters the second quadrant, quasisecular motion of the node is in the opposite direction. Quasisecular disturbances of the parameters are shown as a function of inclination in Table 13. /109

TABLE 13 *

	End of Revolution	Orbit	Inclination i_0						
			90°	63°24′	54°44′	45°	30°	10°	0°
δe	1-	Э	$-0,0555\cdot10^{-7}$	$0,0909\cdot10^{-10}$	-	$-0,0218\cdot10^{-6}$	-	-	$-0,0223\cdot10^{-5}$
		K	$0,0158\cdot10^{-4}$	$0,0638\cdot10^{-7}$	-	$0,0477\cdot10^{-4}$	-	-	$0,0191\cdot10^{-3}$
	10-	Э	$-0,0554\cdot10^{-5}$	$-0,0112\cdot10^{-8}$	-	$-0,0218\cdot10^{-4}$	-	-	$-0,0222\cdot10^{-3}$
		K	$0,0159\cdot10^{-3}$	$0,0619\cdot10^{-6}$	-	$0,0477\cdot10^{-3}$	-	-	$0,0191\cdot10^{-2}$
$\delta\omega$	1-	Э	−14′37″	1″	-	21′42″	-	-	-
		K	−6′40″	0	-	−15′	-	-	−6′
	10-	Э	−2°28′11″	10″	-	3°37′4″	-	-	-
		K	−1°6′43″	0	-	1°57′	-	-	4°9′
$\delta\Omega$	1-	Э	0	−13′12″	−17′2″	−20′51″	−25′30″	−29′28″	−29′26″
		K	0	−11′56″	-	−18′50″	-	-	−26′36″
	10-	Э	0	−2°12′4″	−2°50′22″	−3°28′29″	−4°15′1″	−4°54′24″	−4°54′24″
		K	0	−1°59′20″	-	−3°8′20″	-	- -	−4°26′2″
δt^s	1-	Э	−5,62	−9,23	−11,64	−14,65	−19,15	−15,45	−15,75
		K	−3,65	−6,19	-	−9,68	-	-	−15,52
δr, m	3-	Э	26,4	0,00	11,54	58,02	193,5	378,10	408,06
		K	−0,201	0,00	-	−0,916	-	-	−9,815
Δr, km	3-	Э	0,3	75,9	98,2	120,5	-	-	-
		K	0,2	76,7	-	127,7	-	168,1	-

Note: The perturbations of elliptical orbits are given in lines headed Э (e_0 = 0.499), while the lines headed K give the perturbations of circular orbits. Perturbations $\delta\omega$ of circular orbits are considered in the left-hand neighborhood of the point $2n\pi$ with respect to the initial (in the sense $n \to 0$) position of the line of apsides.

Periodic disturbances are shown as a function of inclination in Figs. 45-53, while Figures 54-60 show the relationship between quasisecular perturbations and inclination.

[1] Disturbance of the line of nodes in an equatorial orbit should be treated as the limit approached by $\delta\Omega$ when $i \to 0$. /108

* Tr. Note: Commas indicate decimal points.

Effect of the Focal Parameter

An increase in the focal parameter (size of the orbit) reduces the amplitude of periodic and quasisecular disturbances. The relationship between periodic disturbances and the size of polar orbits is shown in Table 14 (elliptical orbits) and in Table 15 (circular orbits) for various values of ω_0.

Quasisecular perturbations at the end of the first revolution are shown as a function of the size of polar orbits in Table 16 (elliptical orbits).

TABLE 14*

p_0, km	δp_{min}, km		$\lvert\delta e\rvert_{max} \cdot 10^{-2}$		$\lvert\delta\omega\rvert_{max}$	δt^S		δr_{min}, km		$\lvert\delta V\rvert_{max}$, m/sec
	$\omega_0 = 0$	$\omega_0 = \pi/2$	$\omega_0 = 0$	$\omega_0 = \pi/2$	$\omega_0 = 0$	$\omega_0 = 0$	$\omega_0 = \pi/2$	$\omega_0 = 0$	$\omega_0 = \pi/2$	$\omega_0 = 0$
6639	−20,20	−20,00	0,149	0,200	34°30′35″	−4,20	−4,05	−6,88	−6,61	7,92
9359	−14,32	−15,29	0,0771	0,101	28°00′26″	−3,50	−3,48	−4,83	−4,78	3,33

TABLE 15*

	p_0, km				
	6678	7378	9378	13378	42378
	Distance from the surface of the Earth h, km				
	300	1000	3000	7000	36000
δp_{min}, km	−20,2	−18,12	−14,25	−9,98	−3,15
$\delta e_{max} \cdot 10^2$	0,2123	0,1738	0,1075	0,05276	0,005247
δr_{min}, km	−7,48	−6,78	−5,33	−3,74	−1,18
$\lvert\delta V\rvert_{max}$, m/sec	7,73	6,03	3,31	1,36	0,08

TABLE 16*

p_0, km	δp, m	$\delta e \cdot 10^5$	$\delta\omega$	δt^S	δr, m
6639	0,244	0,2402	−14′48″	−5,62	−119,4
7359	0,04834	0,0004883	−5′00″	−3,50	−42,35

*Tr. Note: Commas indicate decimal points.

Perturbations of eccentricity δe, focal radius δr, time of motion δt, orbital velocity δV, and also of the functions δq and δk for circular orbits at various distances from the Earth may be illustrated by the graphs appended at the end of this section.

Effect of Initial Eccentricity

A change in initial eccentricity has a considerable effect on the form of perturbations of eccentricity and angular distance of the perigee.

This effect on periodic disturbances of orbital elements for the case of satellite motion in a field of model Б is shown in Figures 61-68. The corresponding variants of the initial conditions are given in Table 17.

TABLE 17 *

Variant Number	Initial Parameters						
						Altitude	
	Ω_0	i_0	p_0, km	e_0	ω_0	Apogee h_A, km	Perigee h_Π, km
14	0	90°		0,1358·10⁻³	0	1 000	998
15	0	45°	11216	0,735	0	36000	100

Disturbances of the orbital elements of circular staellites during motion in a field of model Б are shown in Figures 69-75. The corresponding variants of initial conditions are given in Table 18.

At large, e_0, the function $\delta e(u)$ is negative and comparatively monotonic in nature (Fig. 62). When $e_0 = 0$, the function $\delta e(u)$ becomes positive, and shows three clearly pronounced half-waves over the draconic period (see Fig. 69). As the eccentricity decreases, there is somewhat of an increase in the quantity $|\delta e|_{max}$. For instance, for orbits with approximately identical focal parameters and eccentricities respectively equal to 0.002677 and 0, the quantity $|\delta e|_{max}$ is equal to 0.00149 and 0.00165.

A variation in eccentricity has a still more appreciable effect on the periodic perturbations of the line of apsides. The comparatively smooth behavior of the function $\delta\omega(u)$ for an orbit with high eccentricity loses its monotonic character and becomes more and more oscillatory in nature (see Fig. 63). This effect is considerably more noticeable with an increase in

*Tr. Note: Commas indicate decimal points.

TABLE 18 *

Variant Number	Initial Parameters					Altitutde	
	Ω_0	i_0	p_0, km	e_0	ω_0	Apogee h_A, km	Periqee h_Π, km
16	0	90°	6663,553	0	–	300	300
17	0	90°	7363,553	0	–	1000	1000
18	0	90°	9363,553	0	–	3000	3000
19	0	45°	7363,553	0	–	1000	1000
20	0	63°26′	7363,553	0	–	1000	1000
21	0	0	7363,553	0	–	1000	1000
22	0	0	6996,087	0,0499034	0	1000	300

Note: The change in $\delta\omega$ and $\delta\Omega$ for variants 21 and 22 should be taken in the limiting sense ($i_0 \to 0$).

inclination. As eccentricity decreases, the oscillatory nature of the function $\delta\omega(u)$ becomes more pronounced, the amplitude increasing until finally, when $e_0 = 0$, the line of apsides of polar orbits begins to rotate at an average velocity 2.5 times as great as the rate of satellite revolution (see Fig. 70). The sinusoidal (with a slight secular drift) function $\delta\omega(u)$ for equatorial eliptical orbits is transformed to the usual linear function $\delta\omega(u)$ for circular equatorial orbits (see Fig. 70).

Naturally, the described behavior of the relationship $\delta\omega(u)$ gives an increase in $|\delta\omega|_{max}$ with a reduction in e. For parabolic and hyperbolic orbits, perturbation $|\delta\omega|_{max} \leq 2.5'$. For orbits with moderate eccentricities ($e_0 = 0.0499$, $p_0 = 6{,}996$ km), perturbation $|\delta\omega|_{max} = 2°10'$; for orbits with $e_0 = 0.00376$, $p_0 = 6{,}640$ km, perturbation $|\delta\omega|_{max} = 34°.30.5'$; for circular orbits, as we have seen, the line of apsides generally begins to rotate. /114

Perturbations of the line of nodes $\delta\Omega(u)$ and inclination $\delta i(u)$ change only slightly with a variation in eccentricity. As shown by Figures 64 and 65, there is a reduction in the oscillatory behavior of perturbations in components of the Laplace vector $\delta q(u)$ and $\delta k(u)$ with an increase in e_0.

An increase in eccentricity leads to some smoothing of perturbations of the focal parameter $\delta p(u)$ and inclination $\delta i(u)$, changes the form of the function $\delta t(u)$ somewhat (Fig. 66), and raises the value of $|\delta t|_{max}$.

*Tr. Note: Commas indicate decimal points.

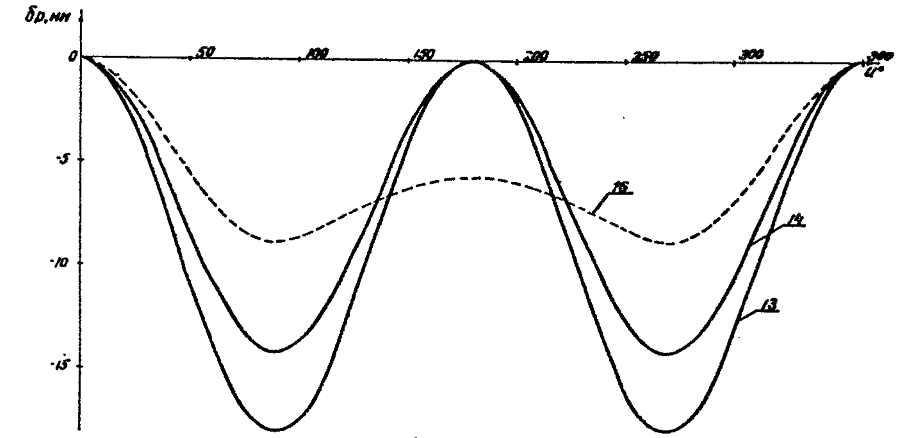


Fig. 61

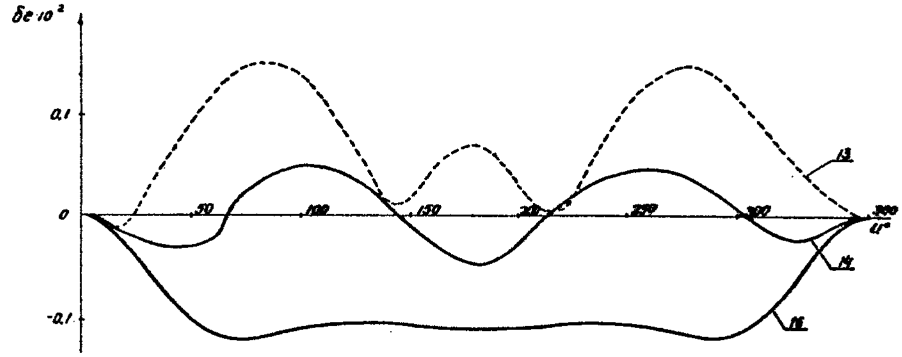


Fig. 62

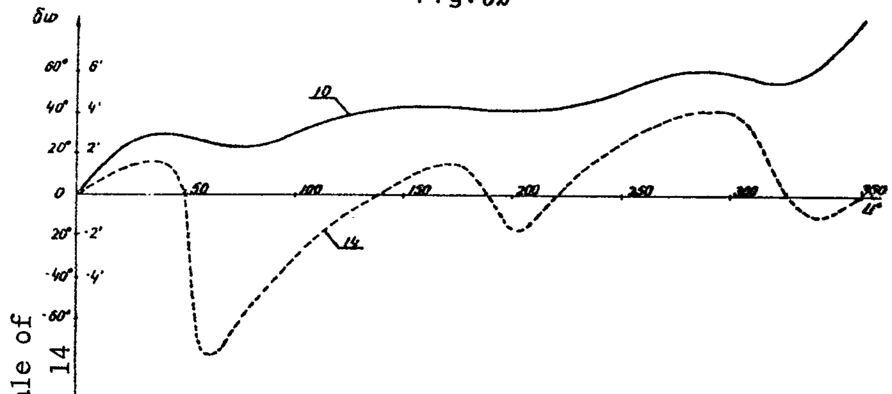


Fig. 63

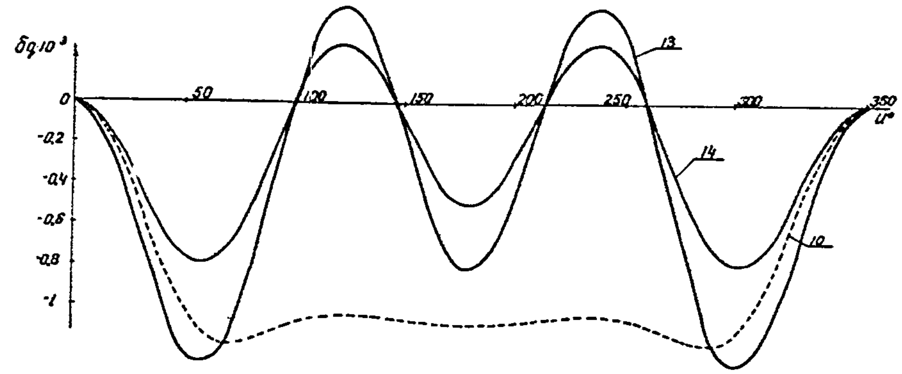


Fig. 64

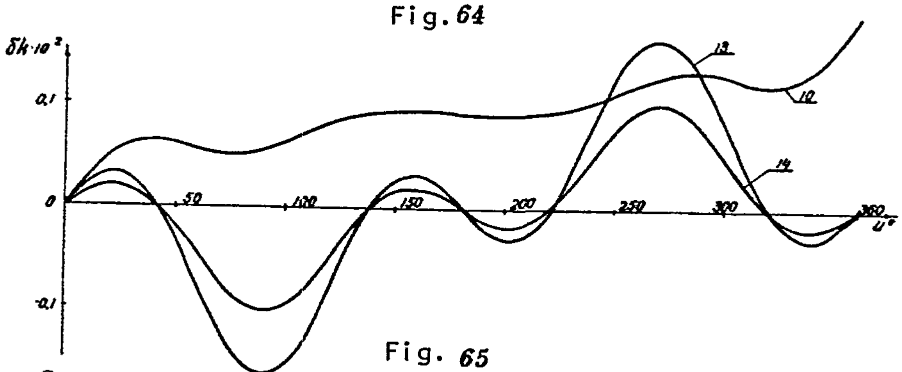


Fig. 65

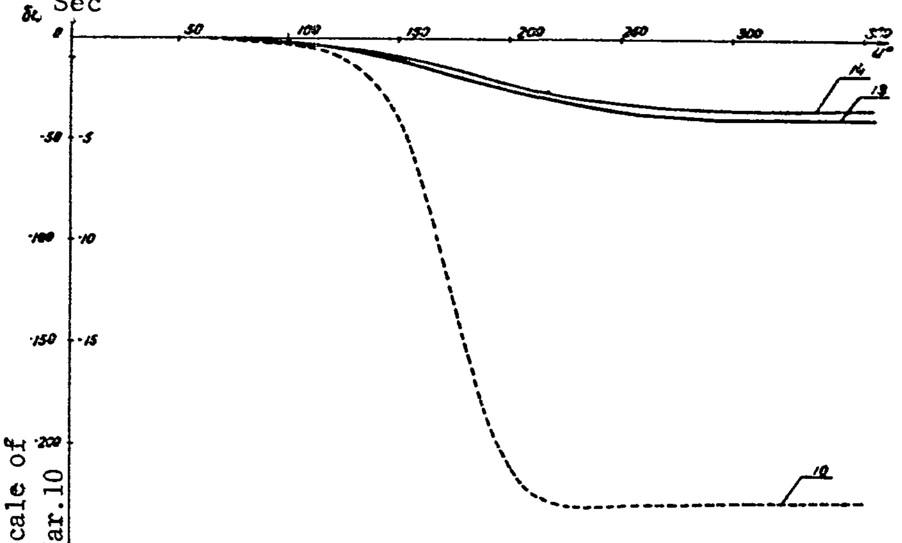


Fig. 66

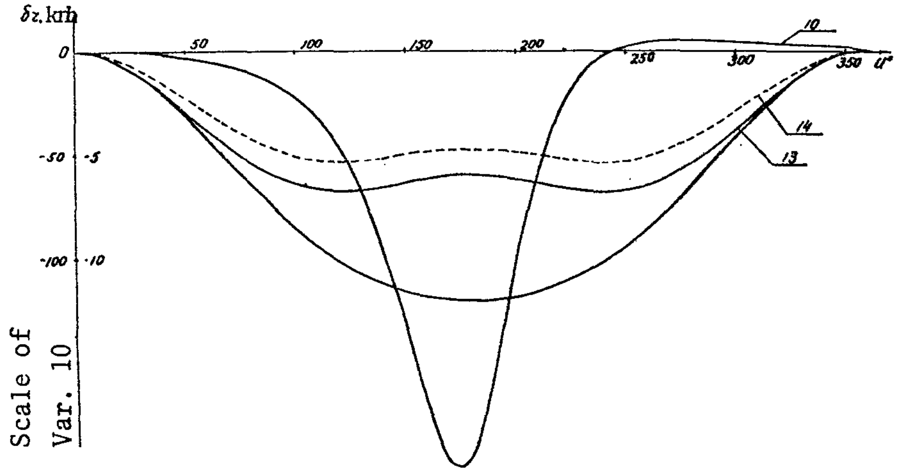


Fig. 67

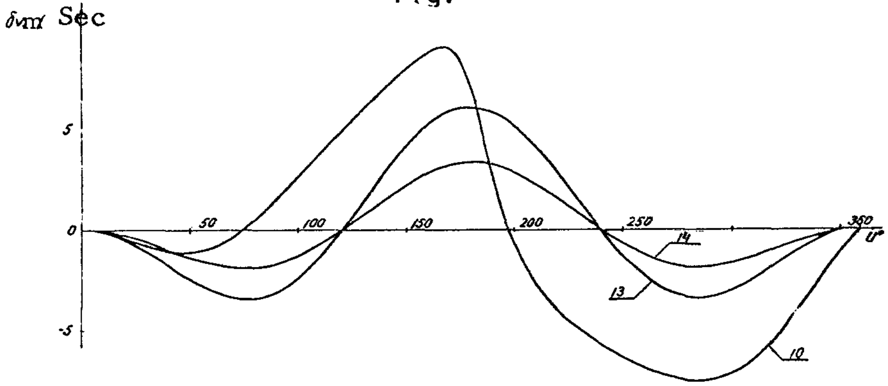
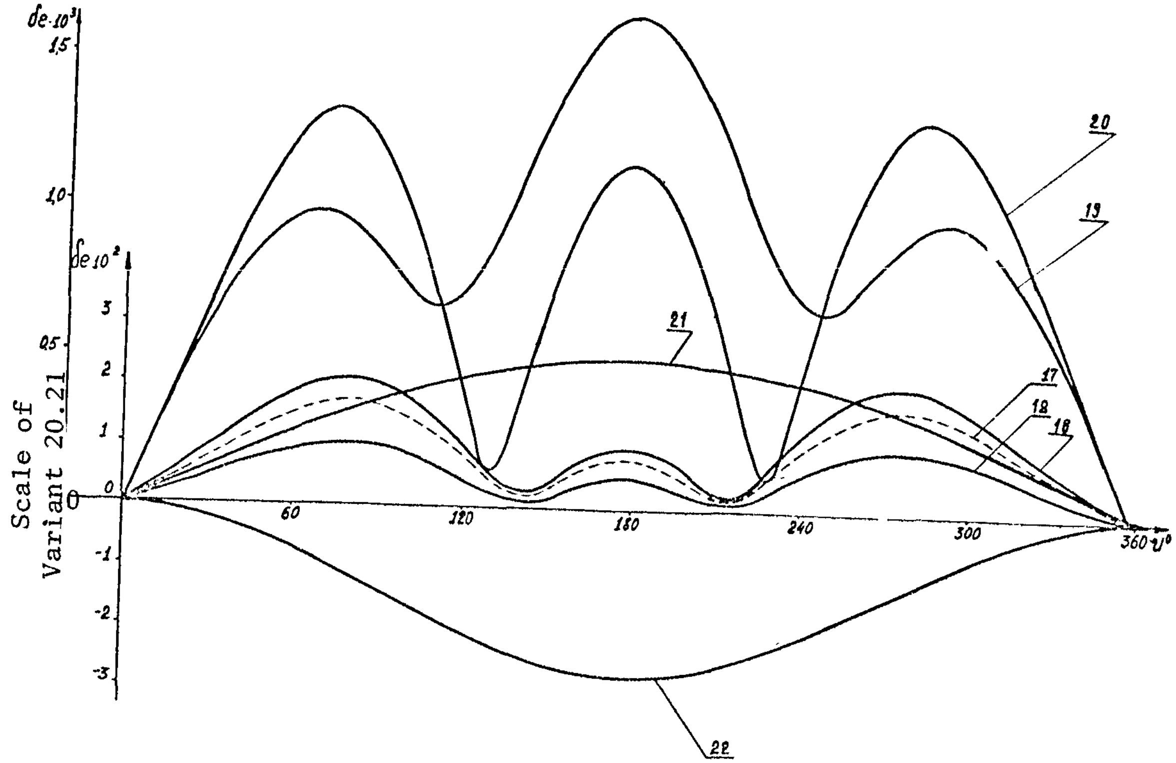


Fig. 68

For instance, for orbits with eccentricities 0, 0.735 and 1.0, these disturbances are equal to $9^s.68$, $-225^s.7$ and $-51{,}320^s$ (the latter value at $u = 170°$); for orbits with identical values of $p_0 \approx 6{,}700$ km, these disturbances come to $-4^s.04$ and $-5^s.60$. /120

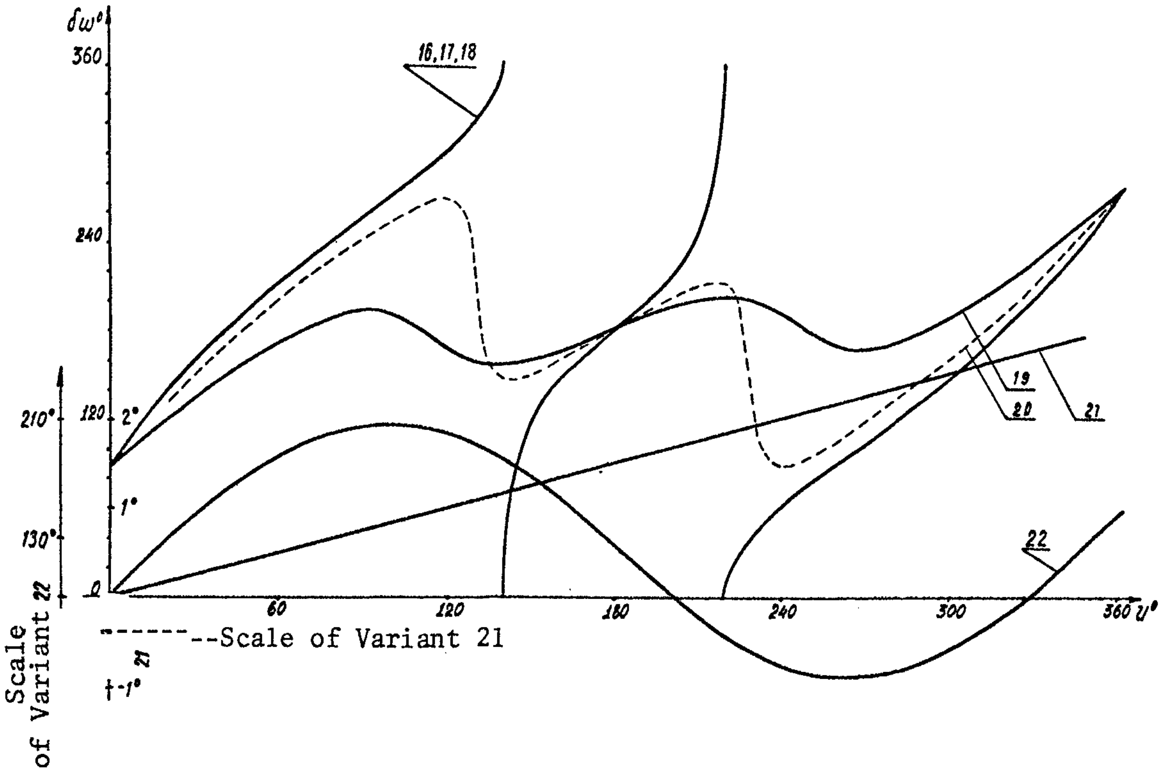
A reduction in eccentricity also leads to an increase in disturbances of the focal radius (Fig. 67), which are especially noticeable for orbits with $e_0 \gtrsim 1$.

The increase in perturbations $\delta t(u)$ and $\delta r(u)$ in orbits with extremely high eccentricities is due to the face that the values of the functions $t(u)$ and $r(u)$ themselves are high in this case.



$\delta e \cdot 10^3$
1,5
1,0
0,5
0
$\delta e \cdot 10^2$
3
2
1
0
-1
-2
-3
60
120
180
240
300
360
υ^0
20
19
21
17
18
16
22
Scale of
Variant 20.21

Fig. 69



δω°
360
240
120
0
16,17,18
19
20
21
22
60
120
180
240
300
360 υ°
210°
130°
2°
1°
-1°
--Scale of Variant 21
Scale of Variant 22

Fig. 70

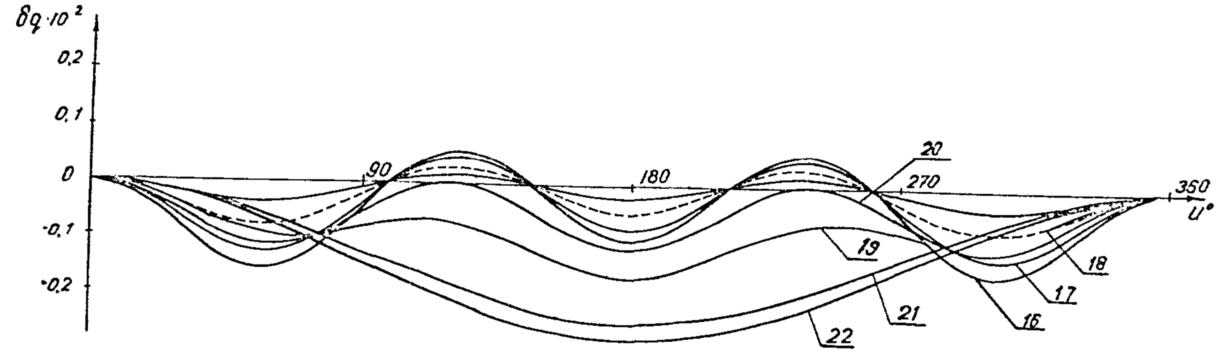


Fig. 71

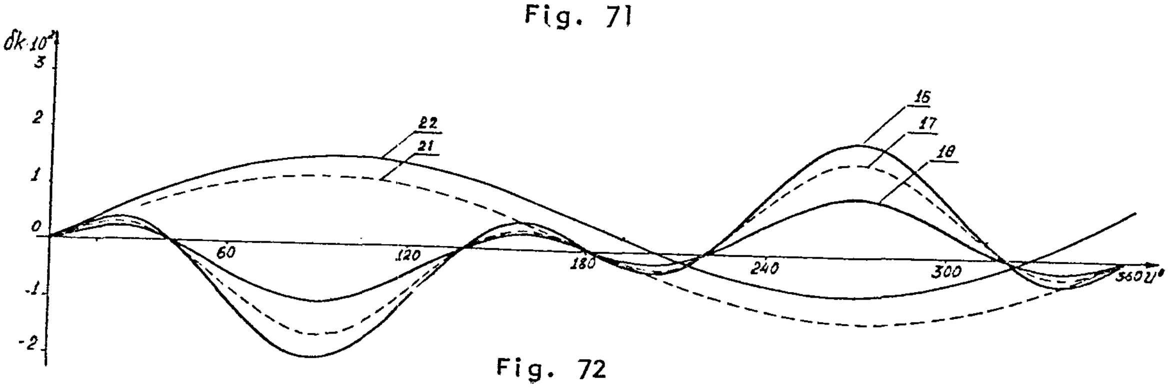


Fig. 72

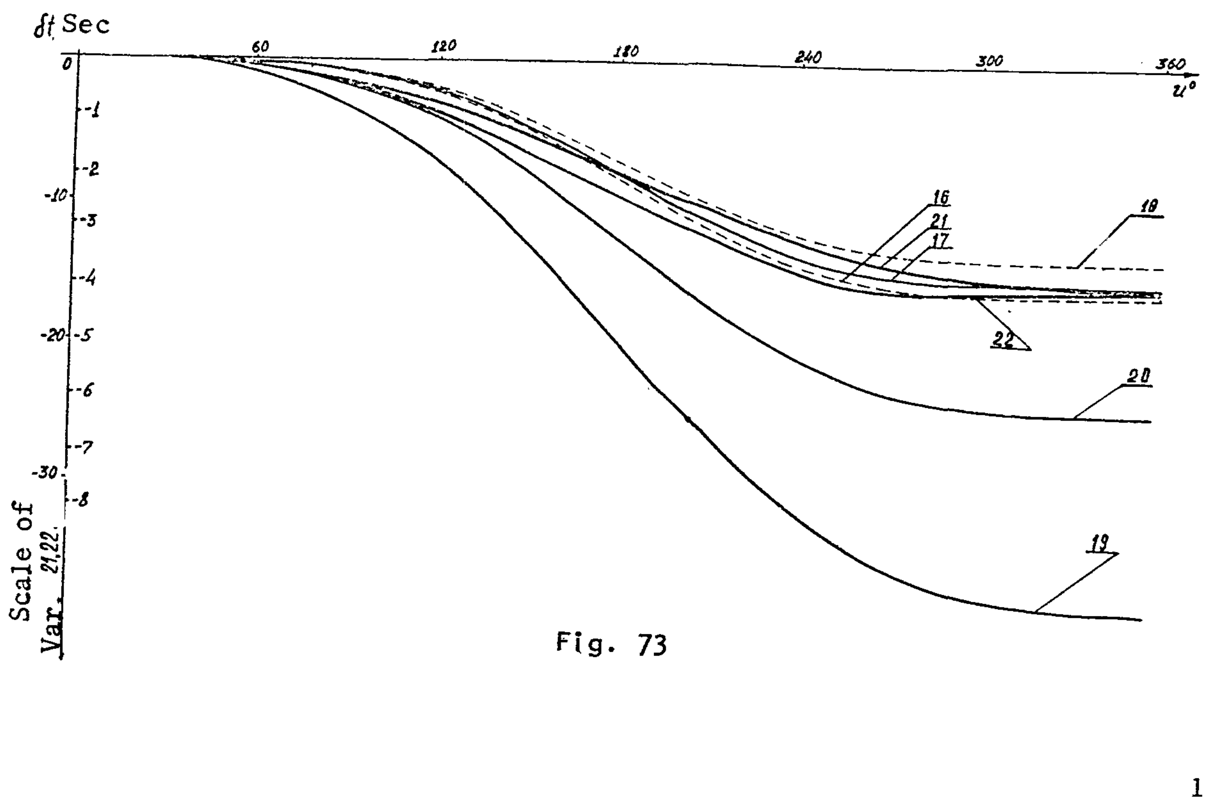


Fig. 73

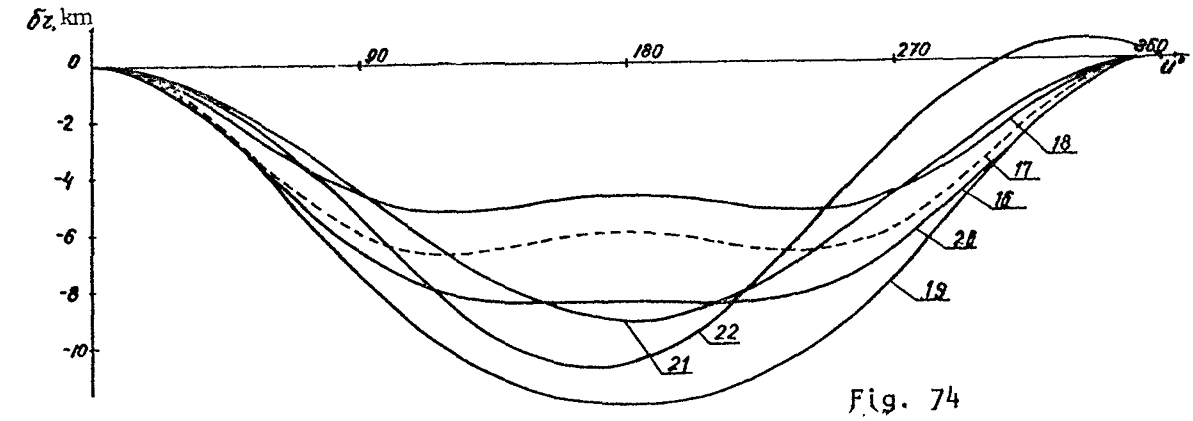


Fig. 74

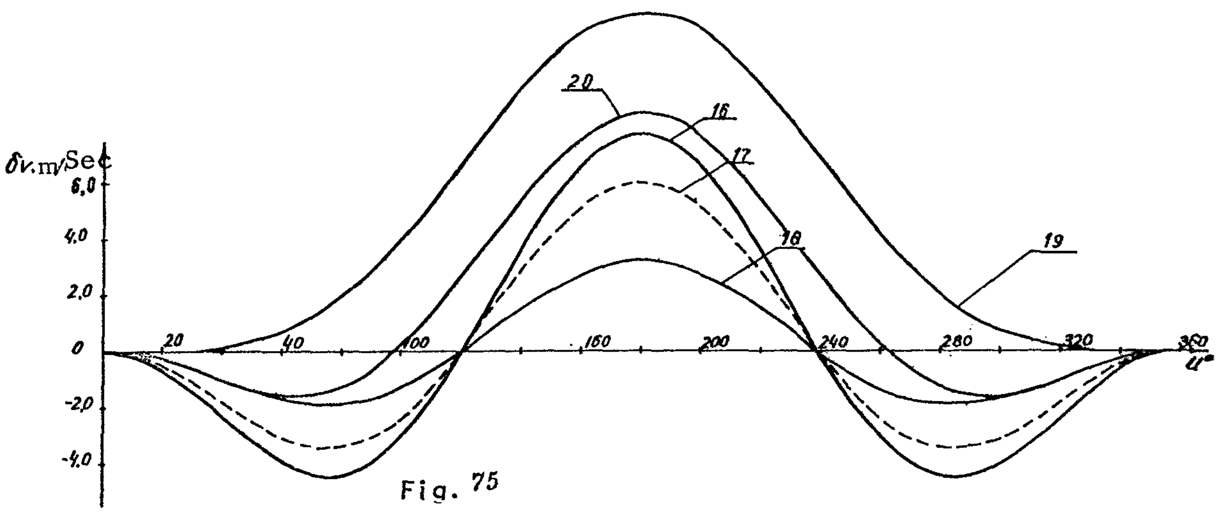


Fig. 75

Not only is there an increase in the value of the perturbation δr, but in the relative quantity $\left|\frac{\delta r}{r}\right|_{max}$ as well. This is shown in Table 19.

TABLE 19*

	p_0, km				
	7378	6996	11216	14730	18400
e_0	0,000	0,0499	0,735	1,000	1,500
$\lvert\delta t\rvert^{s}_{max}$	9,68	–	225,70	51320,00	–
$\left\lvert\frac{\delta r}{r}\right\rvert_{max}$	–	0,00237	0,00477	0,0300	0,0573

*Tr. Note: Commas indicate decimal points.

The perturbation of the orbital velocity $|\delta V|_{max}$ (Figures 68, 75) is somewhat greater for circular than for elliptical orbits: $|\delta V|_{max} = 7.73$ m/sec when $e_0 = 0$; $|\delta V|_{max} = 7.55$ m/sec when $e_0 = 0.0499$; in both cases, $p_0 \approx$ $\approx$ 6,400 km. In general, however, $|\delta V|_{max} \leq 12.2$ m/sec in the interval $0 \leq e_0 \leq 0.735$.

A reduction in eccentricity leads to a considerable increase in the quasisecular perturbations of eccentricity and the line of apsides. The quasisecular perturbations of the line of apsides at the end of the first revolution are shown in Table 20.

TABLE 20*

e_0	0,735	0,0499	0,0003396
$\delta\omega$	8′36″	21′42″	28′24″

TABLE 21*

e_0	p_0, km	δt^S	δr, m
0,1685	7786	−10,11	5,760
0,0499	6996	− 5,62	2,890
0,00378	6639	−	0,234
0,001069	9354	− 3,45	−
0,0003396	7361	− 3,88	−
0,0000000	7400	− 3,85	0,022

This governing principle is violated on circular orbits, for which the secular perturbation $\delta\omega$ cannot be determined because of the indeterminacy of ω at points $2n\pi$ (see Appendix VI). Quasisecular perturbations δr and δt (Table 21) decrease with a reduction in eccentricity, which is explained, as in the case of periodic disturbances, by a reduction in the values of the quantities $\delta r(u)$ and $\delta t(u)$. /121

Effect of the Position of the Perigee ($0 \leq \omega_0 \leq \pi$)

The position of the perigee has its strongest effect on perturbations of the eccentricity and the line of apsides (see Figures 46, 47). As ω_0 is varied over the given interval, periodic perturbations δe change in the fourth

* Tr. Note: Commas indicate decimal points.

decimal place (when $0.0027 \leq e_0 \leq 0.11$). As ω_0 increases, perturbations in $\delta\omega$ become less oscillatory and shift into the upper half-plane.

The effect of the quantity ω_0 on disturbances of the line of apsides $\delta\omega(u)$ is strongly dependent on inclination and eccentricity, increasing with a reduction in i_0 and e_0.

The position of the perigee has almost no effect on the form of periodic disturbances of the remaining functions. Perturbations $\delta p(\omega)$ and $\delta i(\omega)$ are equal at the ends of the interval $[0,\pi/2]$ and have a shallow maximum at the point $\omega_0 = \pi/2$. In this interval, the change in $|\delta\omega|_{max}$ is no greater than 3"; The variation in $|\delta p|_{max}$ is given in Table 22. Also shown there is the reduction in periodic perturbations δt, δr and δV (see Figures 50, 51, 53) with an increase in the angular distance of the perigee.

TABLE 22 *

ω_0	$i_0 = 90°$; $e_0 = 0{,}0499$ $p_0 = 6996$ km				$i_0 = 45°$; $e_0 = 0{,}0499$ $p_0 = 6996$ km			$i_0 = 90°$; $e_0 = 0{,}002677$ $p_0 = 9359$ km		
	$\lvert\delta p\rvert_{max}$, km	$\lvert\delta t\rvert^s_{max}$,	$\lvert\delta r\rvert_{max}$, km	$\lvert\delta V\rvert_{max}$, m/sec	$\lvert\delta p\rvert_{max}$, km	$\lvert\delta t\rvert^s_{max}$,	$\lvert\delta r\rvert_{max}$, km	$\lvert\delta p\rvert_{max}$, km	$\lvert\delta t\rvert^s_{max}$	$\lvert\delta r\rvert_{max}$, km
0	19,71	5,60	8,00	7,55	9,84	14,60	14,88	14,32	3,50	4,83
45°	20,20	5,14	5,17	6,72	9,90	14,42	15,00	15,29	3,48	4,78
90°	19,71	4,00	5,55	5,99	9,84	13,95	13,86	–	–	–

As the quantity ω_0 passes into the second quadrant, the perturbation of eccentricity $\delta e(\omega)$ changes sign, while disturbances in the angular distance of the perigee $\delta\omega(\omega)$ and in the time of motion $\delta t(\omega)$ alter their values somewhat (e.g. $|\delta t|_{max}$ changes by about $0^s.9$). Perturbations of the quantities $\delta p(\omega)$, $\delta\Omega(\omega)$ and $\delta i(\omega)$ are symmetric with respect to the axis $\omega_0 = \pi/2$.

Quasisecular disturbances of all functions, with the exception of δt and $\delta\Omega$, increase (in absolute value) with an increase in ω_0 in the interval $[0,\pi/2]$.

Perturbation δt in this case decreases, while $\delta\Omega$ remains unchanged. /122
All this is shown in Table 23, where the values of the quasisecular perturbances at the end of the tenth revolution are given.

---

* Tr. Note: Commas indicate decimal points.

TABLE 23 *

		$\omega_0 = 0°$	$\omega_0 = 45°$	$\omega_0 = 90°$
δp, km	1	−0,58	+18,71	+24,27
	2	−0,62	−14,34	−20,74
$\delta e \cdot 10^3$	1	0	0,01676	0,01970
	2	0	−0,0480	−0,0582
$\delta \omega$	1	−2°26′11″	−2°26′46″	−2°27′55″
	2	3°37′4″	3°38′42″	3°41′55″
δt^S	1	−56,18	−51,41	−40,50
	2	−146,48	−144,40	−139,70
δr, m	1	282,3	−9669	−14963
	2	644,40	15333	22547

Note: The numeral 1 indicates an orbit with the following parameters: $i_0 = 90°$; $e_0 = 0.0499$; $p_0 = 6{,}996$ km. The numeral 2 indicates an orbit with the following parameters: $i_0 = 45°$; $e_0 = 0.0499$; $p_0 = 6{,}996$ km.

The analysis given above may be generalized in the form of the graphs given in Figures 78-86 which show the effect of initial parameters on perturbations of orbital elements for satellite motion in the gravitational field of a spheroid.

§9. Effect of Orbit Parameters on Disturbances of Satellite Motion due to Gravitational Anomalies

Gravitational anomalies are taken as those described in §5 and §7, acting on a satellite moving in the field of the terrestrial spheroid.

The effect of initial conditions on periodic and quasisecular perturbations of orbital elements due to gravitational anomalies is shown in Figures 87-94. The corresponding variants of initial conditions are given in Table 24.

TABLE 24*

Variant Number	Initial Parameters							
	Ω_0'	i_0	p_0, km	e_0	ω_0	u_0	h_A, km	h_Π, km
23	0	90°	7786,10	0,1685	0	0	3000	300
24	0	90°	9358,49	0,002677	0	0	3000	2950

* Tr. Note: Commas indicate decimal points.

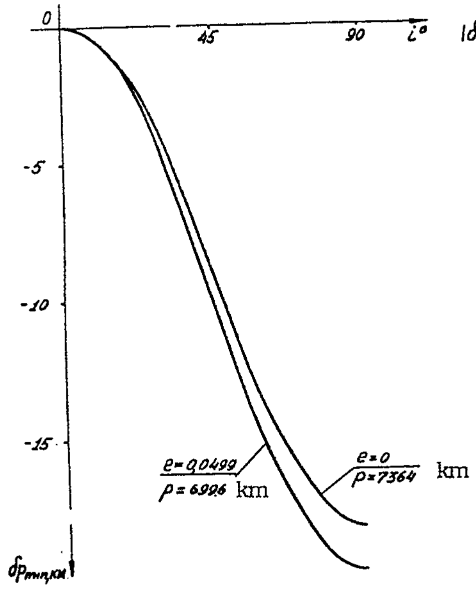


Fig. 76

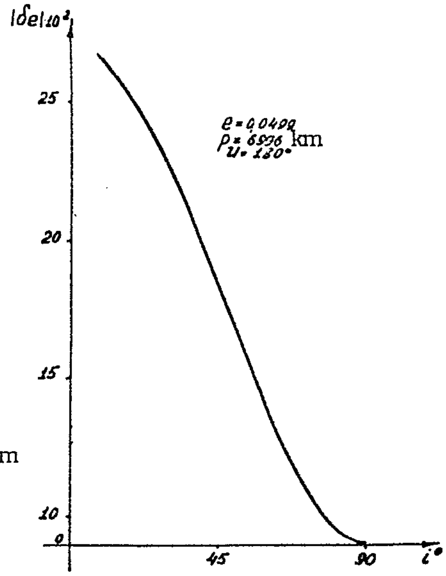


Fig. 77

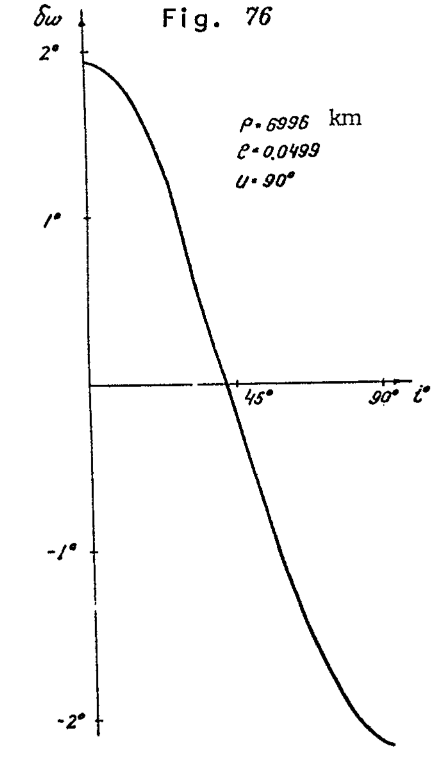


Fig. 78

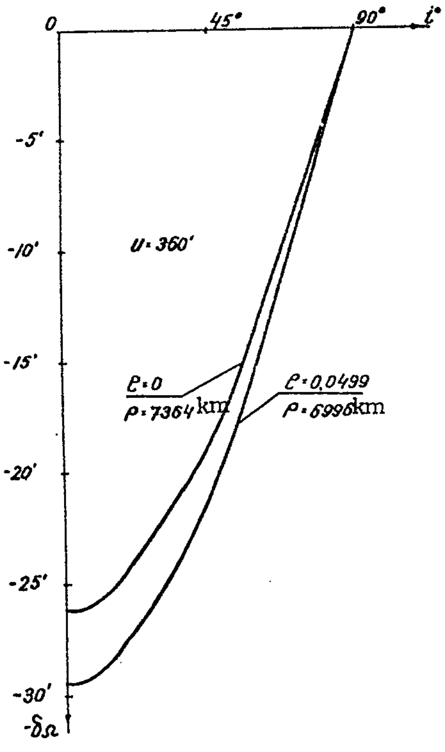


Fig. 79

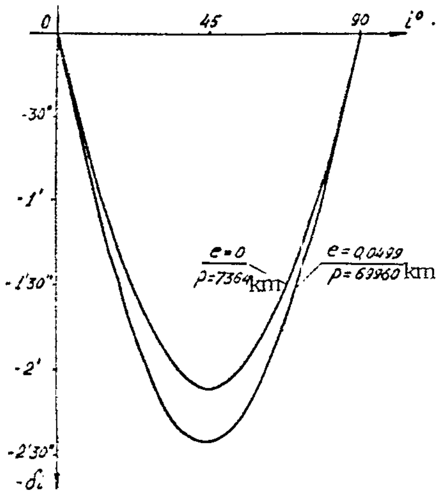


Fig. 80

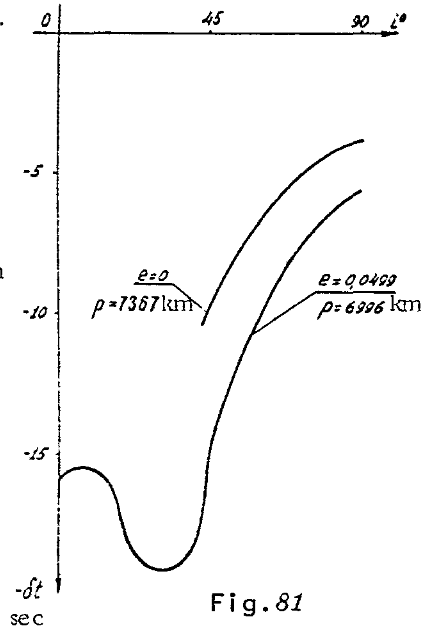


Fig. 81

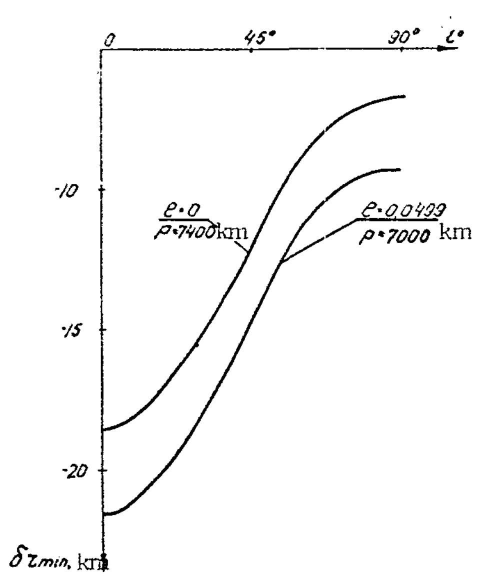


Fig. 82

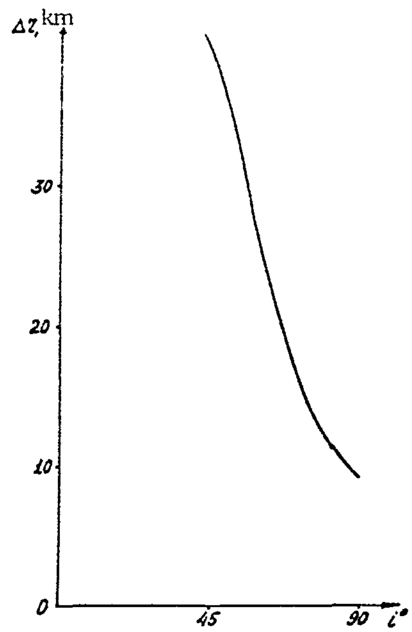


Fig. 83

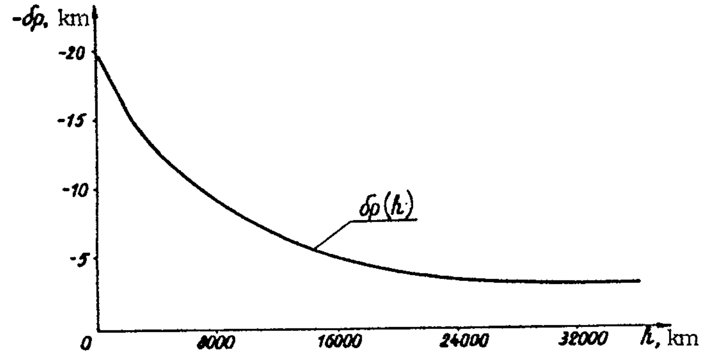


Fig. 84

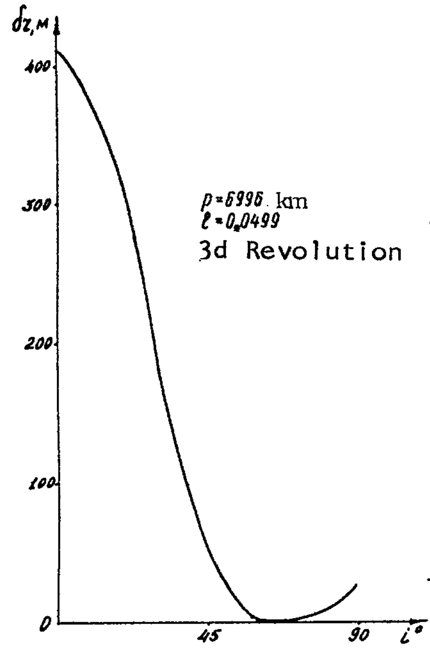


Fig. 85

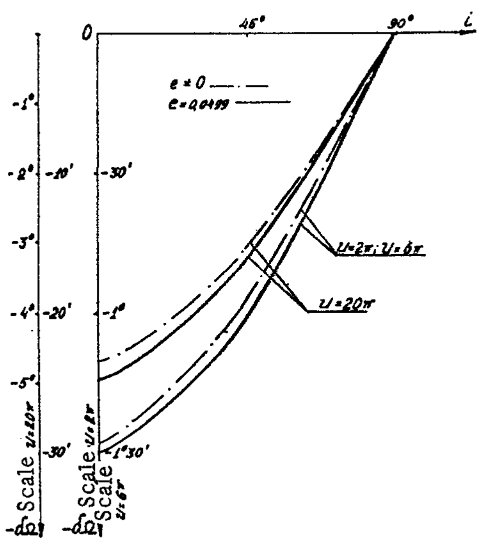


Fig. 86

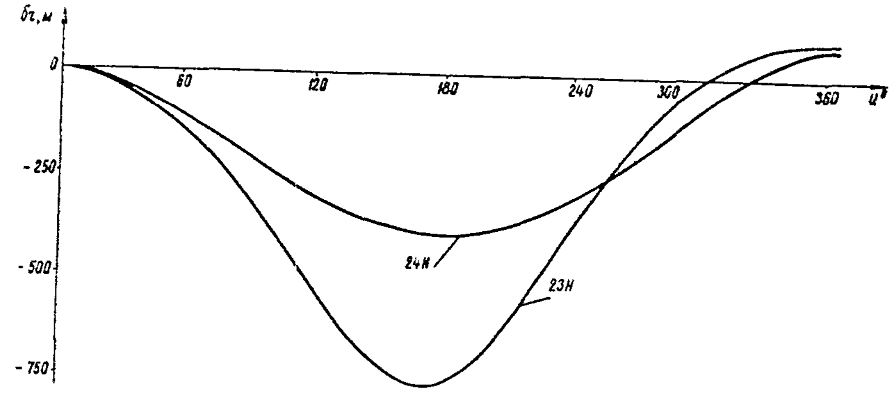


Fig. 87

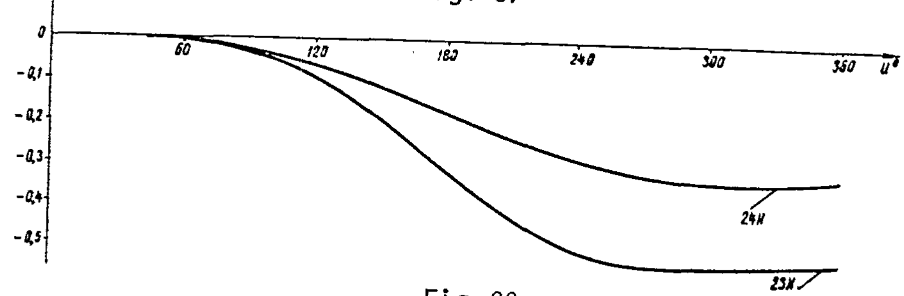


Fig. 88

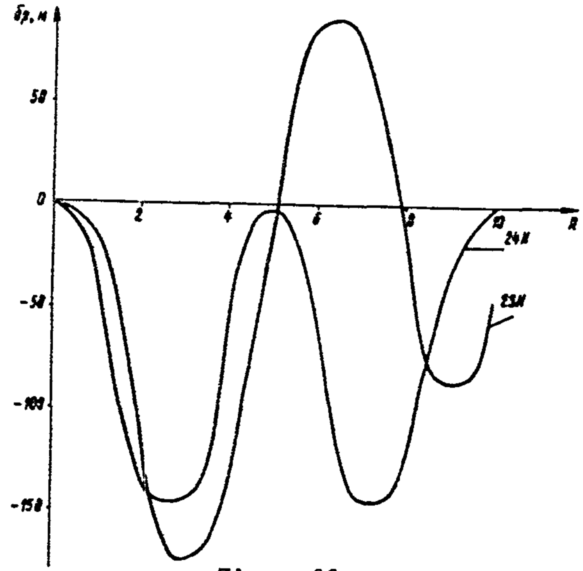


Fig. 89

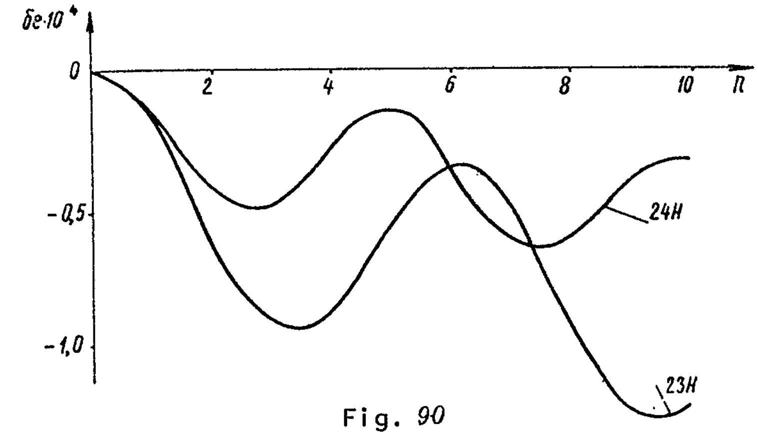


Fig. 90

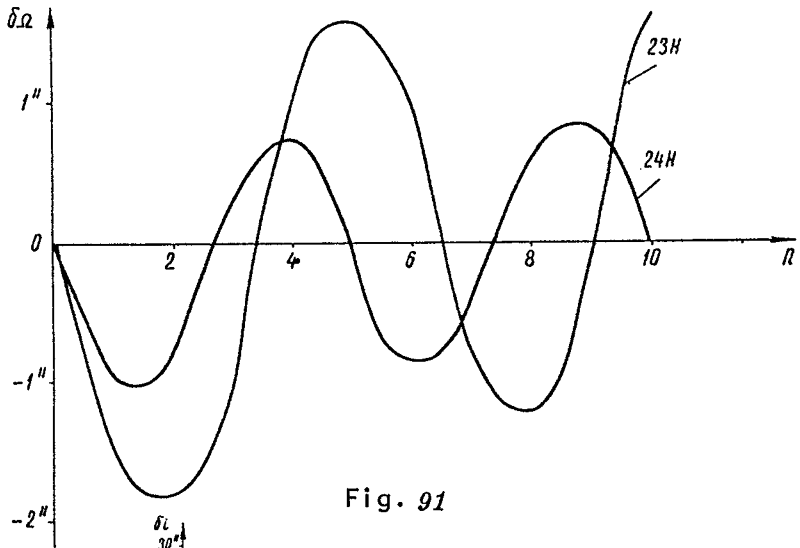


Fig. 91

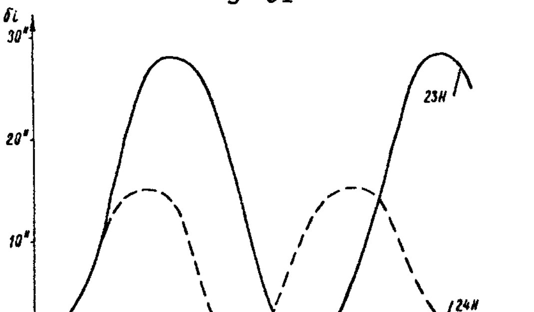


Fig. 92

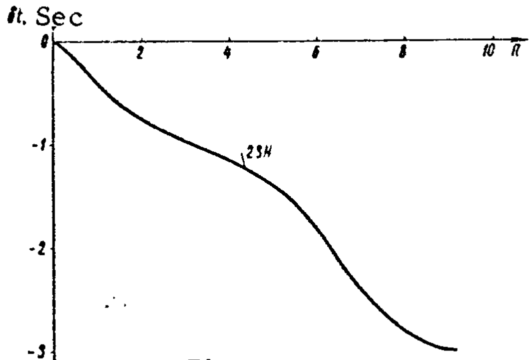


Fig. 93

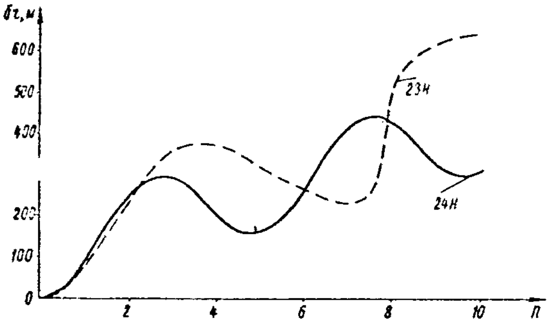


Fig. 94

Effect of Inclination on Periodic Disturbances /128

As inclination is reduced from 90° to 63.4°, perturbations of the focal parameter from the square of flattening increase somewhat, and then decrease as the inclination is further reduced. Disturbances of the focal parameter δp due to asymmetry decrease monotonically, while perturbations from triaxiality and the overall effect of anomalies increase.

A change in inclinations has the strongest effect on perturbations δp from triaxiality (and from the sum total of anomalies, which change by about 200 meters as the inclination is varied over the range $[0, \pi/2]$.

Eccentricity perturbations from each of the anomalies individually increase with a reduction in the inclination. Perturbations from the sum total of the anomalies have a shallow minimum at $i_0 =$ $= 63.4°$. A change in inclination has the strongest effect on perturbations of eccentricity from triaxiality and from the sum total of the anomalies, which do not exceed $0.04 \cdot 10^{-3}$. /129

Disturbances in the motion of the line of apsides decrease with a reduction in inclination from 90° to 63.4°. The greatest change results from the effect of the square of flattening and asymmetry, and reaches a value of 30" -35".

Disturbances from gravitational anomalies in the longitude of the ascending node increase with a reduction in inclination. Decreased inclination has its strongest effect on perturbations from the sum total of the anomalies (change in perturbations $\delta\Omega$ by about 13'.5).

Perturbations of orbital inclination due to anomalies have a maximum at $i_0 = 63.4°$. Disturbances in the inclination of equatorial orbits from overall anomalies are not equal to zero, but reach an amplitude of 1"-1.5".

As the inclination decreases, there is an increase in perturbations of the function δt due to the square of flattening and asymmetry. Perturbations due to triaxiality and the sum total of the anomalies have a maximum at $i_0 = 63.4°$ where the value of $|\delta t|$ increases by about $0^s.2$. Perturbations δr or in all cases have a minimum at $i_0 = 63.4°$ (the value of $|\delta r|$ decreases by about 500 meters). The same thing applies to the function Δr. However, the minimum in this case is not so pronounced for perturbation from all anomalies, with the exception of perturbations from triaxiality. In the latter case, perturbations Δr when $i_0 = 0$ are not as great as when $i_0^* = 63.4°$, in view of the pronounced maximum of the function δi at an inclination of 63.4°.

Effect of Inclination on Quasisecular Disturbances

Quasisecular disturbances δp have a maximum when $i_0 = i_0^* = 63.4°$ in all cases except for disturbances from asymmetry, where disturbances of the parameter increase with a reduction in inclination. At the same time, perturbations increase for eccentricity and the longitude of the ascending node. Perturbations of the line of apsides have a minimum at $i_0 = i_0^*$, while disturbances of the inclination from triaxiality and overall anomalies show a maximum at the same value of inclination (disturbances from the square of flattening and asymmetry are equal to zero at all inclinations).

Perturbations of the functions δt due to the square of flattening and asymmetry have a minimum at $i_0 = i_0^*$, while disturbances from triaxiality and overall anomalies have a maximum at this same point. The change in δt from these anomalies may reach 3^s with a change in inclination over the interval $[0,\pi/2]$. The same thing applies to the function δr, which may change by as /130
much as 700 meters. The function Δr has a minimum at $i_0 = 63.4°$.

Effect of Eccentricity on Periodic Perturbations

Perturbations of the focal parameter δp and eccentricity δe due to anomalies decrease in all cases with a reduction in eccentricity. The maximum change in $|\delta p|$ and $|\delta e|$ reaches several meters and $0.4 \cdot 10^{-5}$, respectively, as the eccentricity is varied over the range $0 \leq e_0 \leq 0.05$. Perturbations of the line of apsides $\delta\omega$ increase appreciably with a reduction in eccentricity, particularly perturbations from triaxiality, asymmetry and overall anomalies (by about 75°). Perturbations in the inclination due to the square of flattening and asymmetry increase with a reduction in eccentricity. Disturbances of the inclination due to triaxiality do not change with a variation in eccentricity. The effect of anomalies on perturbations in the longitude of the ascending node as well as in the function δt and δr decreases with a reduction in eccentricity (by no more than $0^s.1$ and 100 meters, respectively, for the last two functions).

Effect of Eccentricity on Quasisecular Disturbances

The effect of anomalies on disturbances in the focal parameter, the line of apsides and eccentricity increases with a reduction in eccentricity. Perturbations of the inclination due to the square of flattening and triaxiality increase, while those due to overall anomalies decrease. Perturbations in the line of nodes in these cases behave inversely.

The inclination, longitude of the ascending node and function δt do not undergo any quasisecular perturbations due to asymmetry of the hemispheres. In the remaining cases, perturbations of δt, just as perturbations of δr, increase with a reduction in eccentricity.

By the end of the tenth revolution, the greatest changes (for $0 \leq e_0 \leq$ ≤ 0.05) in the functions δt and δr are from triaxiality and overall anomalies. These changes do not exceed 1^s and 1,200 meters, respectively, in absolute value.

The effect which the angular distance of the perigee has on periodic disturbances due to gravitational anomalies shows up in a change in perturbations of the focal parameter δp and radius δr by about 300 meters as well as in a change in perturbations of time δt by about $0^s.2$.

For quasisecular disturbances, this effect shows up in a change in perturbations of the radius δr by about 50 meters, perturbations of the time /131
of motion δt by about $1^s.5$, and in a change in perturbations of the longitude of the ascending node Ω and inclination δi by about 9" (all indicated values are reached at the end of the tenth revolution).

The effect which inclination, eccentricity and the angular distance of the perigee have on the maximum (in absolute value) periodic disturbances is shown in Tables 25, 26 and 27, respectively, while the effect of these elements on quasisecular perturbations is shown in Tables 28, 29 and 30.

The numerals I, II, III and IV in Table 25 designate orbits having the following initial parameters:

$$\begin{array}{lllll} \mathrm{I} & - \; i_0 = 90^\circ; & \omega_0 = 0; & p_0 = 6996 \text{ km}; \\ \mathrm{II} & - \; i_0 = 63^\circ,4; & \omega_0 = 0; & p_0 = 6996 \text{ km}; \\ \mathrm{III} & - \; i_0 = 63^\circ,4; & & p_0 = 7400 \text{ km}; \\ \mathrm{IV} & - \; i_0 = 10^\circ; & & p_0 = 7400 \text{ km}; \end{array}$$

The result of comparison of the perturbations of orbits I-II and III- /132
IV characterizes the effect which changes in inclination have on perturbations in elliptical and circular orbits due to anomalies.

TABLE 25 *

		$\lvert\delta p\rvert$, m	$\lvert\delta e\rvert\cdot 10^5$	$\lvert\delta\omega\rvert$	$\lvert\delta\Omega\rvert$	$\lvert\delta i\rvert$	$\lvert\delta t\rvert^S$	$\lvert\delta r\rvert$, m	Δr, m
		Perturbations from the square of flattening							
$e_0 = 0{,}05$	I	27	0	40″	–	–	–	47	–
	II	30	0,37	8″	2″	–	0,01	–	–
$e_0 = 0$	III	23,5	0,24	42′14″	1″,81	2″,75	0,004	12,3	65
	IV	3,20	0,67	20″	9″	0″,25	0,04	48	323
		Perturbations from triaxiality							
$e_0 = 0{,}05$	I	220	6,5	1′30″	–	–	0,24	550	–
	II	250	6,7	1′10″	9″,5	3″	0,42	530	–
$e_0 = 0$	III	246	6,3	72°7′10″	3″,75	3″	0,33	473	354
	IV	409	10,3	94°5′15″	7″,5	0″,84	0,22	957	284
		Perturbations from asymmetry							
$e_0 = 0{,}05$	I	47	0	25″	–	–	–	45	–
	II	23	0,30	15″	1″,8	0	0,009	–	–
$e_0 = 0$	III	19	0,27	74°7′	1″,51	2″,75	0,0078	9,4	44
	IV	10,5	0,30	14°5′	9″	0″,83	0,04	23	263
		Perturbations from the sum total of the anomalies							
$e_0 = 0{,}05$	I	230	7,5	1′26″	–	–	0,26	620	890
	II	.270	7,0	1′17″	7″	3″	0,42	550	–
$e_0 = 0$	III	233	6,5	82°7′26″	1″,86	3″	0,33	473	300
	IV	420	9,9	95°00′34″	13″,5	1″,22	0,30	908	603

TABLE 26*

	$\lvert\delta p\rvert$, m	$\lvert\delta e\rvert\cdot 10^4$	$\lvert\delta\omega\rvert$	$\lvert\delta\Omega\rvert$	$\lvert\delta i\rvert$	$\lvert\delta t\rvert^S$	$\lvert\delta r\rvert$, m	Δr, m
	Perturbations from the square of flattening							
I	30	0,037	8″	2″	0	0,010	–	–
II	23,5	0,024	42′14″	1″,81	2″,75	0,004	12,3	65
	Perturbations from triaxiality							
I	250	0,67	1′10″	9″,5	3″	0,42	530	–
II	246	0,63	72°7′10″	3″,75	3″	0,33	473	354
	Perturbations from asymmetry							
I	23	0,030	15″	1″,8	0	0,009	–	–
II	19	0,027	74°7′	1″,51	2″,75	0,008	9,4	44
	Perturbations from the sum total of the anomalies							
I	261	0,65	1′17″	7″	3″	0,420	550	–
II	253	0,65	82°7′26″	1″,86	3″	0,332	473	300

*Tr. Note: Commas indicate decimal points.

The numeral I in Table 26 designates an orbit with parameters $e_0 = 0.05$, $\omega_0 = 0$, $p_0 = 6{,}996$ km, $i_0 = 63.4°$, while the numeral II indicates parameters $e_0 = 0$, $p_0 = 7{,}400$ km, $i_0 = 63.4°$.

A comparison of variants I and II shows the effect which a change in eccentricity has on disturbance due to gravitational anomalies.

TABLE 27*

		$\lvert\delta p\rvert$, m	$\lvert\delta e\rvert \cdot 10^4$	$\lvert\delta\omega\rvert$	$\lvert\delta t\rvert^s$	$\lvert\delta r\rvert$, m	Δr, m
$i_0 = 80°$	I	230	0,75	1′26″	0,28	620	980
	II	220	0,50	6′	0,43	600	–
$i_0 = 45°$	I	270	0,50	18′	0,18	750	–
	III	280	0,60	3′	0,30	450	–

Given in Table 27 are perturbations in the orbital parameters from the sum total of the anomalies. The numerals I, II and III designate orbits with parameters:

I – $e_0 = 0{,}05$; $\omega_0 = 0$; $p_0 = 6996$ km;
II – $e_0 = 0{,}05$; $\omega_0 = 90°$; $p_0 = 6996$ km;
III – $e_0 = 0{,}05$; $\omega_0 = 45°$; $p_0 = 6996$ km;

The result of comparison shows the effect which a change in the angular distance of the perigee has on perturbation from gravitational anomalies at various inclinations of the orbit. /133

In Tables 28 and 29, the disturbances of elliptical orbits are considered over a period of 10 revolutions, while those of circular orbits are considered over 5 revolutions. The maximum absolute values of the perturbations are indicated in all cases.

The numerals I, II, III and IV in Table 28 designates orbits with the following parameters:

I – $i_0 = 80°$; $\omega_0 = 0$; $p_0 = 6996$ km;
II – $i_0 = 63°{,}4$; $\omega_0 = 0$; $p_0 = 6996$ km;
III – $i_0 = 63°{,}4$; $p_0 = 7400$ km;
IV – $i_0 = 10°$; $p_0 = 7400$ km;

*Tr. Note: Commas indicate decimal points.

TABLE 28*

		$\lvert\delta p\rvert$, m	$\lvert\delta e\rvert\cdot 10^3$	$\lvert\delta\omega\rvert$	$\lvert\delta\Omega\rvert$	$\lvert\delta i\rvert$	$\lvert\delta t\rvert^s$	$\lvert\delta r\rvert$, m	Δr, m
		Perturbations from the square of flattening							
$e_0 = 0,05$	I	–	–	30″	–	–	0,339	–	--
	II	–	–	28″	21″	–	–	–	--
$e_0 = 0$	III	0,0093	$0,33\cdot 10^{-4}$	–	9″,5	0	0,021	0,0093	310
	IV	0,0060	$0,86\cdot 10^{-3}$	–	46″	0	0,430	0,3400	1616
		Perturbations from triaxiality							
$e_0 = 0,05$	I	–	–	28″	–	–	2,058	180,77	--
	II	274	0,073	20″	21″	21″	2,36	200,00	--
$e_0 = 0$	III	809	0,075	–	4″	27″,5	3,23	1355,00	144
	IV	71	0,15	–	9″	3″,17	0,36	1340,00	300
		Perturbations from asymmetry							
$e_0 = 0,05$	I	–	–	6″	–	–	–	551,000	--
	II	–	$0,154\cdot 10^{-3}$	1″	0	–	–	–	--
$e_0 = 0$	III	0,0073	$0,97\cdot 10^{-4}$	–	0	0	0,00	0,939	3
	IV	0,4120	0,00278	–	0	0	0,01	165,000	10
		Perturbations from the sum total of the anomalies							
$e_0 = 0,05$	I	–	–	48″	–	–	2,34	885,3	--
	II	274	0,0724	8″,3	11″	21″	2,46	199,0	--
$e_0 = 0$	III	809	0,0748	–	7″,62	25″,4	3,21	1355,0	263
	IV	72	0,142	–	54″	3″,4	0,12	1274,0	1881

Comparison by pairs of the disturbances of orbits I and II, and also those of orbits III and IV shows the effect which inclination has on quasi-secular perturbations from anomalies for elliptical and circular orbits, respectively.

The numerals I and II in Table 29 designate orbits with the following parameters:

$$\mathrm{I} - e_0 = 0,05;\ \omega_0 = 0;\ p_0 = 6996\ \text{km};\ i_0 = 63^\circ,4;$$
$$\mathrm{II} - e_0 = 0;\ p_0 = 7400\ \text{km};\ i_0 = 33^\circ,4.$$

A comparison of these data reveals the effect of eccentricity on the quasisecular disturbances from gravitational anomalies.

Table 30 summarizes the disturbances of the orbital parameters from the sum total of the anomalies. The numerals I, II, and III denote orbits with the following parameters:

$$\mathrm{I} - e_0 = 0,05;\ \omega_0 = 0;\ p_0 = 6996\ \text{km};$$
$$\mathrm{II} - e_0 = 0,05;\ \omega_0 = 90^\circ;\ p_0 = 6996\ \text{km};$$
$$\mathrm{III} - e_0 = 0,05;\ \omega_0 = 45^\circ;\ p_0 = 6996\ \text{km};$$

A comparison of these perturbations shows how a change in the angular distance of the perigee affects perturbations from anomalies at various inclinations of the orbit.

* Tr. Note: Commas indicate decimal points.

TABLE 29*

	$\lvert\delta p\rvert$, m	$\lvert\delta e\rvert \cdot 10^4$	$\lvert\delta\omega\rvert$	$\lvert\delta\Omega\rvert$	$\lvert\delta i\rvert$	$\lvert\delta t\rvert^S$	$\lvert\delta r\rvert$, m	Δr, m
	Perturbations from the square of flattening							
I	–	–	28″	21″	–	–	–	–
II	0,0093	0,33×10⁻³	–	0	9″,5	0,021	0,0093	310
	Perturbations from triaxiality							
I	274	0,73	20″	21″	21″	2,36	200	–
II	246	0,75	–	4″	28″	3,23	1355	144
	Perturbations from asymmetry							
I	–	0,0015	1″	0	–	–	–	–
II	0,0073	0,0272	–	0	0	0	0,939	3
	Perturbations from the sum total of the anomalies							
I	274	0,72	8″,3	11″	21″	2,46	199	–
II	246	0,75	–	7″,6	25″	3,21	1355	283

TABLE 30*

		$\lvert\delta p\rvert$, m	$\lvert\delta e\rvert \cdot 10^4$	$\lvert\delta\omega\rvert$	$\lvert\delta\Omega\rvert$	$\lvert\delta i\rvert$	$\lvert\delta t\rvert^S$	$\lvert\delta r\rvert$, m	Δr, m
$i_0 = 90°$	I	–	–	48′	0″	0	2,340	885	885
	II	183,7	0,05118	7′30″	2″	44″	1,947	912	–
$i_0 = 45°$	I	899,6	0,32400	3′35″	18″	6″	0,395	732	315
	III	609,3	0,32000	3′34″	9″	14″	1,658	772	–

§10. Effect of Orbit Parameters on Disturbances of Satellite Motion in the Field of a Triaxial Ellipsoid /135

A change in the position of the line of nodes has the strongest effect on the amplitude and form of periodic disturbances in the field of a triaxial ellipsoid (see Figures 95-106 and Table 31).

The nature of functions δp, δe, δt, δr and $\delta\omega$ changes comparatively little. As the longitude Ω of the ascending node increases, there is somewhat of an increase in the absolute value of the first four functions, and a more appreciable increase in $\delta\omega$. The change in perturbations of the parameters of motion with a change in the longitude of the ascending node over

* Tr. Note: Commas indicate decimal points.

TABLE 31*

Variant Number	Initial Parameters							
	Ω_0	i_0	p_0, km	e_0	ω_0	u_0	h_A, km	h_Π, km
$1\Gamma_1$	0	90°	6996	0,0499	0	0	1000	300
$1\Gamma_3$	45°	90°	6996	0,0499	0	0	1000	300
$1\Gamma_2$	90°	90°	6996	0,0499	0	0	1000	300

the range $0 \leq \Omega_0 \leq \pi/2$ is limited to the values: $|\delta p|_{max} \leq 0.7$ km; $|\delta e|_{max} \leq$ $\leq 0.11 \cdot 10^{-3}$; $|\delta\omega|_{max} \leq 24'.5$ (the last inequality is written for $e_0 = 0.004$; it increases with a reduction in e_0 and i_0); $|\delta i| \leq 18''$; $|\delta\Omega| \leq 6''$; $|\delta t| \leq$ $\leq 0^s.82$; $|\delta r| \leq 0.95$.

The nature of perturbations in the inclination of the orbit and in the longitude of the ascending node is most strongly affected by a change in Ω_0. (see Figures 95 and 96).

Shown in Figures 99-101 are the maximum disturbances in the parameters of a polar orbit as a function of the longitude of the ascending node. The periodic perturbations in the inclination of a polar orbit during motion in the field of an ellipsoid may be close to zero at some Ω_0

The effect which the initial orbital parameters have on periodic disturbances of the elements at various values of Ω_0 is shown in Table 32, where the figures represent the greatest change in disturbances of the corresponding parameter as Ω_0 is varied over the interval $[0,\pi/2]$.

The numerals I, II, III and IV in Table 32 designate orbits having the following parameters:

$$
\begin{aligned}
&\text{I} - i_0 = \pi/2;\ p_0 = 6996\text{ km}\ e_0 = 0{,}0499;\ \omega_0 = 0;\\
&\text{II} - i_0 = \pi/2;\ p_0 = 6996\text{ km};\ e_0 = 0{,}0499;\ \omega_0 = \pi/2;\\
&\text{III} - i_0 = \pi/4;\ p_0 = 6996\text{ km};\ e_0 = 0{,}0499;\ \omega_0 = 0;\\
&\text{IV} - i_0 = \pi/4;\ p_0 = 6639\text{ km};\ e_0 = 0{,}003765;\ \omega_0 = 0.
\end{aligned}
$$

As may be seen from Table 32, a change in parameters ω_0, e_0 and i_0 mainly affects the relationship between the longitude of the ascending node and perturbations of the function $\delta\omega$, and in part δe. This effect is extremely small for functions Δr and δt (particularly for polar orbits). Thus, for the change indicated in Table 32 for parameters ω_0, e_0, i_0 and Ω_0, short-period /136

* Tr. Note: Commas indicate decimal points.

TABLE 32*

	I	II	III	IV
$\lvert\delta p\rvert_{max}$, km	0,500	0,423	0,448	0,420
$\lvert\delta e\rvert_{max}\cdot 10^3$	0,066	0,018	0,114	0,150
$\lvert\delta\omega\rvert_{max}$	6′22″	4′17″	18′33″	24′33″
$\lvert\delta i\rvert_{max}$	6″	–	4″	0
$\lvert\delta\Omega\rvert_{max}$	–	–	1″	0
$\lvert\delta t\rvert^{s}_{max}$	0,800	0,750	–	0,780
$\lvert\delta r\rvert_{max}$, km	0,900	0,762	–	0,800

disturbances Δr_{max} and $|\delta t|_{max}$ will differ by no more than 300 meters and $0^s.05$, respectively.

Quasisecular disturbances of the elements δp, δe, δi and $\delta\Omega$ (which undergo diurnal long-period oscillations, as was pointed out in §8) are symmetric with respect to the horizontal axis (the axis of the number of revolutions of the satellite n) with a variation in Ω_0 over the interval $[0,\pi/2]$. Within this interval, the maximum values of the quantities $|\delta p|$, $|\delta e|$ and $|\delta i|$ are lower, and that of $|\delta\Omega|$ is higher, than at the ends of the interval. For the given change in Ω_0, the maximum changes in parameters do not exceed the following values: $|\delta p| \leq 730$ m; $|\delta e| \leq 0.16\cdot 10^{-3}$; $|\delta\omega| \leq 5'.2$; $|\delta\Omega| \leq 22''$; $|\delta i| \leq 2'$; $|\delta t| \leq 3^s.45$ (the last inequality is written for an eccentricity $e_0 = 0.003765$; the given value of δt is attained at the end of the tenth revolution).

Thus the maximum possible overall (quasisecular and periodic) deviations of the elements of the orbit with a variation in the longitude of the ascending node in the field of a triaxial ellipsoid do not exceed the following values: $|\delta p| \leq 1.5$ km; $|\delta e| \leq 0.27\cdot 10^{-3}$; $|\delta\omega| \leq 30'$; $|\delta\Omega| \leq 28''$; $|\delta i| \leq 2'.3$; $|\delta t| \leq 4^s.3$ (the last value is attained at the end of the tenth revolution, as was pointed out above); $|\delta r| \leq 1.6$ km.

Shown in Table 33 is the effect of orbital parameters on the relationship between quasisecular perturbations of the elements and variation of the longitude of the ascending node in the field of a triaxial ellipsoid.

* Tr. Note: Commas indicate decimal points.

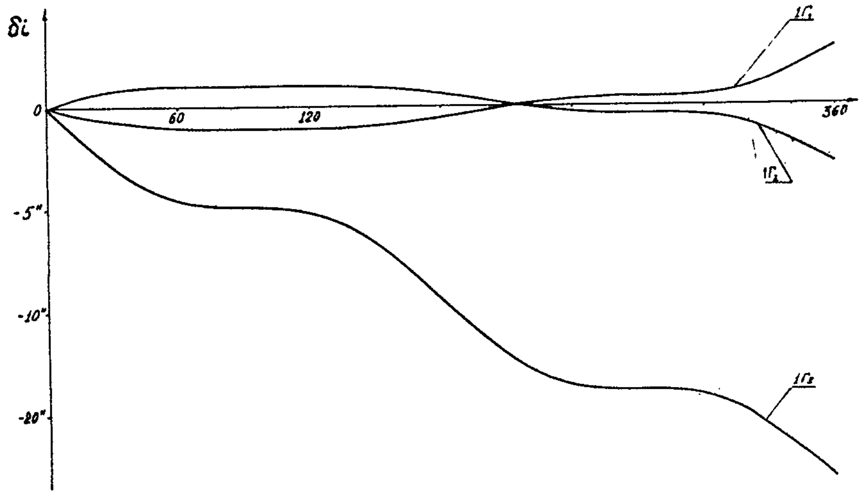


Fig. 95

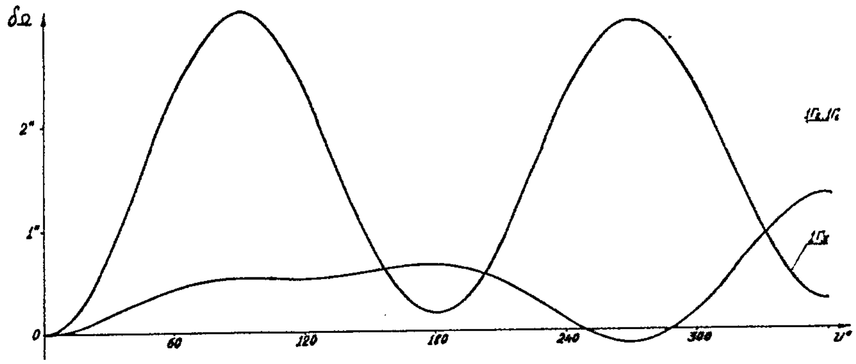


Fig. 96

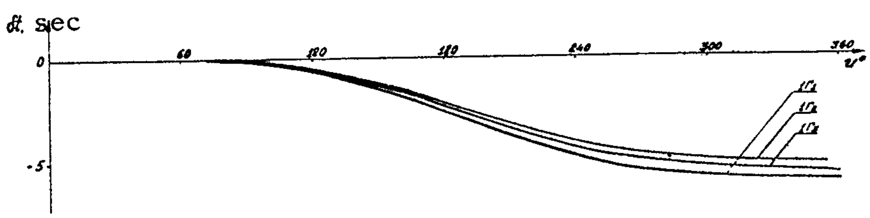
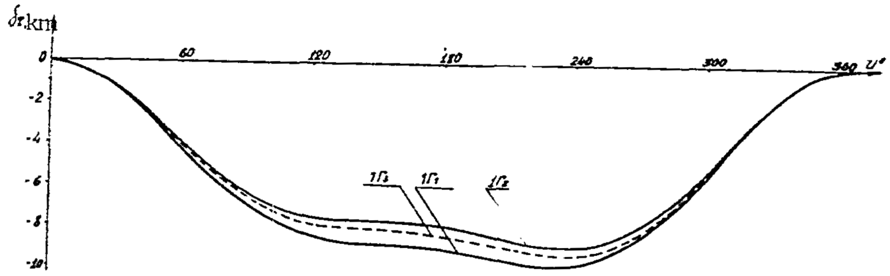


Fig. 97



Fiq. 98

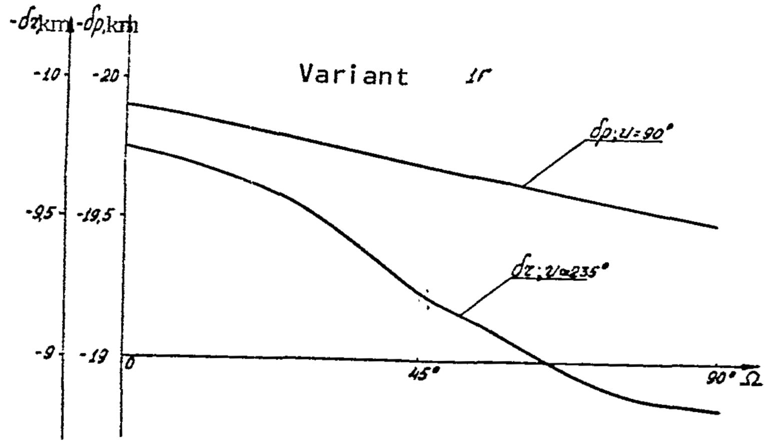


Fig. 99

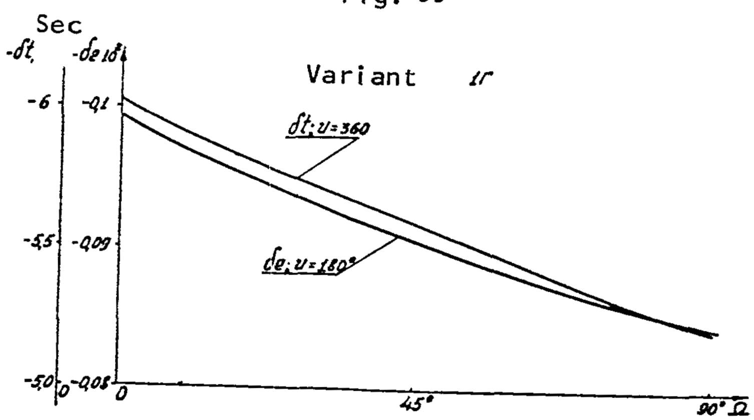


Fig. 100

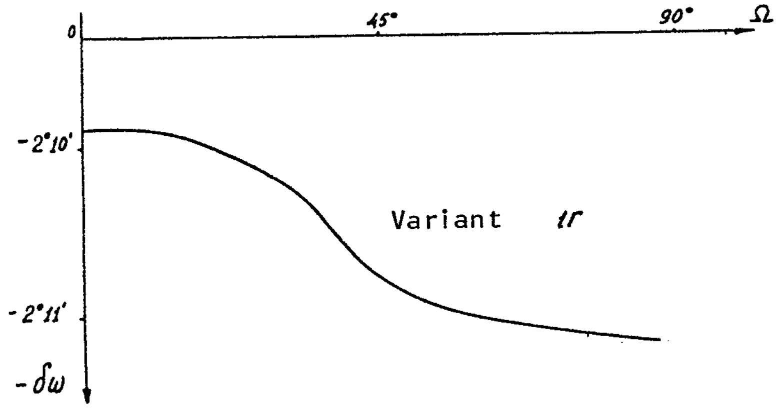


Fig. 101

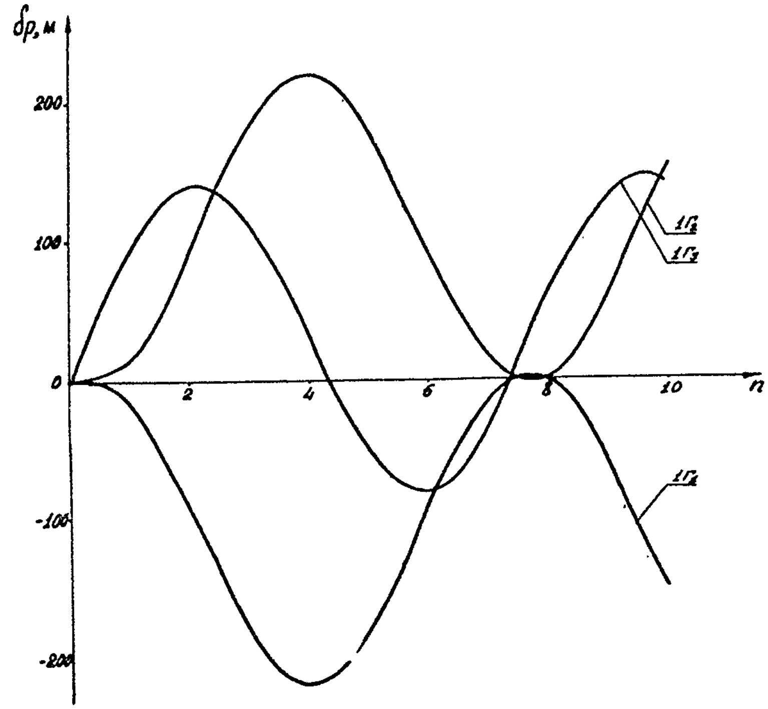


Fig. 102

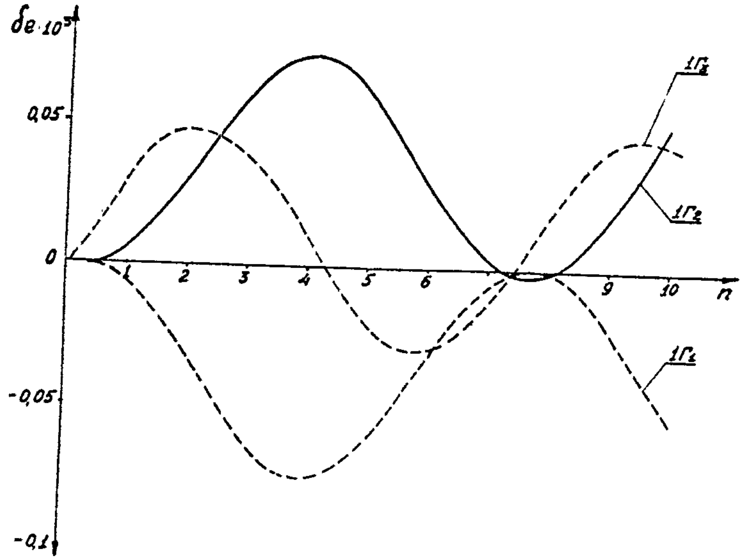


Fig. 103

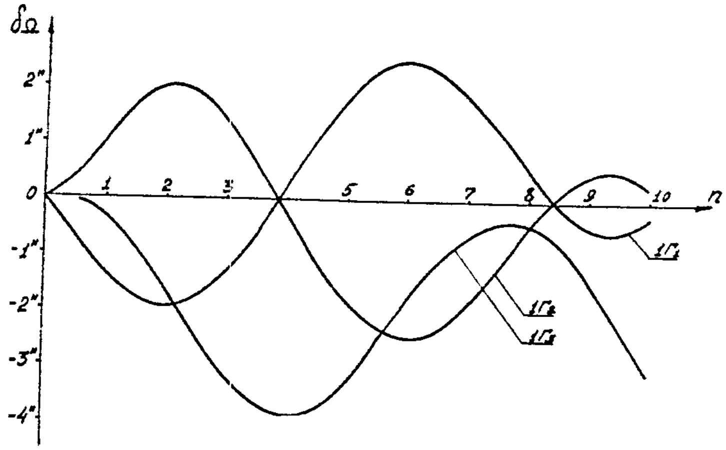


Fig. 104

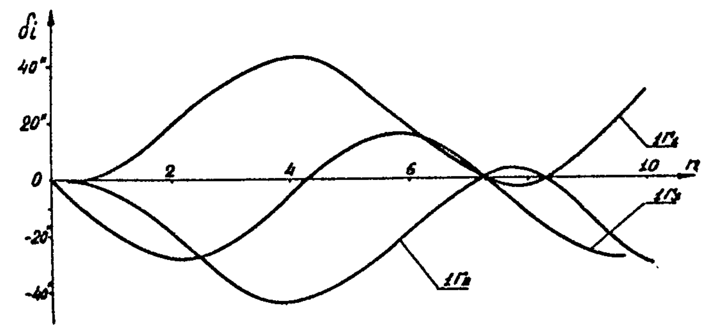


Fig. *105*

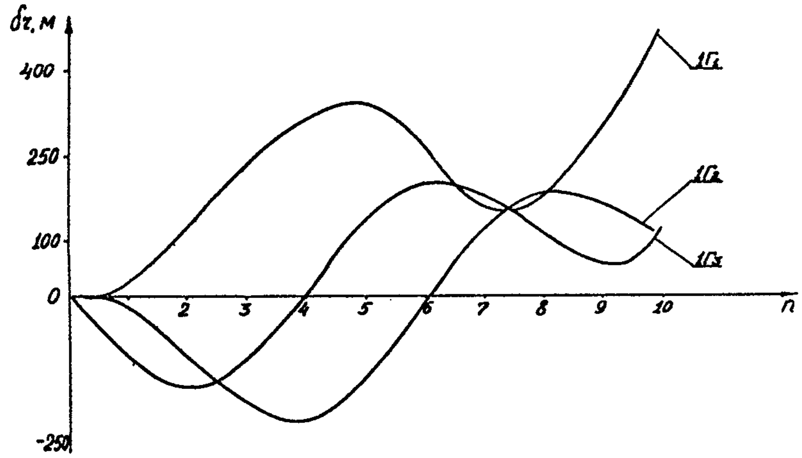


Fig. 106

TABLE 33 *

	I	II	III	IV
$\|\delta p\|$, km	0,675	1,147	0,420	0,426
$\|\delta e\| \cdot 10^3$	0,109	0,095	0,57	0,15
$\|\delta\omega\|$	3′ 20″	5′22″	50″	5′14″
$\|\delta i\|$	13″	28″	5″	6″
$\|\delta\Omega\|$	28″	10″	40″	1′42″
$\|\delta t\|^s$	0,9	3,37	4,0	3,45
$\|\delta r\|$, km	0,168	1,680	0,500	0,540

The numerals I, II, III and IV in Table 33 designate orbits with the following parameters:

$$\begin{aligned}
&\mathrm{I} - i_0 = \pi/4;\ p_0 = 6996 \text{ km};\ e_0 = 0{,}0499; \quad \omega_0 = 0; \\
&\mathrm{II} - i_0 = \pi/4;\ p_0 = 6996 \text{ km};\ e_0 = 0{,}0499; \quad \omega_0 = \pi/4; \\
&\mathrm{III} - i_0 = \pi/2;\ p_0 = 6996 \text{ km};\ e_0 = 0{,}0499; \quad \omega_0 = 0; \\
&\mathrm{IV} - i_0 = \pi/2;\ p_0 = 6639 \text{ km};\ e_0 = 0{,}003765; \quad \omega_0 = 0.
\end{aligned}$$ *

As may be seen, the position of the perigee, inclination of the orbit and amount of eccentricity have a strong effect on the relationship between Ω_0 and perturbations of the line of apsides $\delta\omega$. The position of the perigee, in addition, has a strong effect on perturbations of the functions δt and δr, and orbital inclination has a strong effect on perturbation of the function δt.

A change in the inclination of the orbit does not basically change the nature of the perturbations of parameters for elliptical motion. The resultant difference in the amplitudes of the disturbances is limited to the following figures (for a range of variation in inclination $45° \leq i_0 \leq 90°$):

$|\delta p| \leq 150$ m; $|\delta e| \leq 0.6 \cdot 10^{-4}$; $|\delta\omega| \leq 50''$; $|\delta\Omega| \leq 5''$; $|\delta i| \leq 17''$; $|\delta t| \leq 0^s.5$; $|\delta r| \leq 0.5$ km.

As was pointed out above, long-period perturbations due to diurnal rotation of the Earth show up in quasisecular disturbances of a number of parameters. As the inclination is reduced, the amplitude of oscillations of this type in the focal parameter increases[1], while the amplitude of oscillations in the inclination decreases. The amplitude of long-period oscillations of the functions δe and δr and quasisecular perturbations of the line of apsides have a maximum at $i_0 = 63.4°$.

---

[1] In the field of a triaxial ellipsoid, the diurnal motion of the Earth has a greater effect on long-period disturbances of the focal parameter than motion of the line of apsides. Therefore, the nature of the change in long-period perturbations with a change in inclination in the given case is not the same as in the field of a spheroid. /142

* Commas indicate decimal points.

The effect of inclination on short-period perturbations of some parameters is shown in Table 34.

TABLE 34*

		Model	Inclination		
			$i_0 = 90°$	$i_0 = 63°,4$	$i_0 = 45°$
δp_{min}, km	$e_0 = 0{,}0499$	Б	−18,71	−15,76	−9,84
		E	−19,83	−16,00	−10,10
	$e_0 = 0{,}0000$	Б	−18,12	−14,48	−9,06
		E	–	−14,72	–
$\delta e \cdot 10^2$	$e_0 = 0{,}0499$	Б	−0,09110	−0,1280	−0,1820
		E	−0,09836	−0,1348	−0,1875
$u = 180°$	$e_0 = 0{,}0000$	Б	0,17380	0,1161	0,1646
		E	–	0,1212	–
$\|\delta\omega\|_{max}$	$e_0 = 0{,}0499$	Б	2°9′53″	1°22′38″	10′12″
		E	2°8′7″	1°22′21″	9′55″
$\delta\Omega_{min}$	$e_0 = 0{,}0499$	Б	0	−13′12″	−20′51″
		E	−1″,33	−13′19″	−20′54″
	$e_0 = 0{,}0000$	Б	0	−11′56″	−18′50″
		E	–	−11′55″	–
δi_{min}	$e_0 = 0{,}0499$	Б	0	−2′17″ 1)	−2′25″
		E	2″,9	−2′00″ 1)	−2′27″
	$e_0 = 0{,}0000$	Б	0	−1′42″	−2′7″
		E	–	−1′44″	–
δt^S_{min}	$e_0 = 0{,}0499$	Б	−5,6	−9,20	−14,70
		E	−6,05	−9,66	−14,74
	$e_0 = 0{,}0000$	Б	−3,85	−7,74	−9,68
		E	–	−7,98	–
δr_{min}, km	$e_0 = 0{,}0499$	Б	−9,27	−10,96	−14,78
		E	−9,75	−11,50	−14,57
	$e_0 = 0{,}0000$	Б	−6,80	−8,49	−12,10
		E	–	−8,94	–
Δr_{max}, km	$e_0 = 0{,}0499$	Б	9,27	25,293	40,126
		E	9,27	25,507	–
	$e_0 = 0{,}0000$	Б	–	25,56	–
		E	–	25,52	–

The letters Б and E correspond to the two models of the gravitational field described in §6.

A change in eccentricity during motion in the field of a triaxial ellipsoid has the same effect on periodic and quasisecular disturbances as during motion in a spheroidal field. Periodic perturbation of the line of apsides $\delta\omega$ undergoes the strongest change (an increase) with a reduction in

1) $i_0 = 54°44'$. *Tr. Note: Commas indicate decimal points.

eccentricity. For a change in eccentricity to 0.003, the difference in perturbation $|\delta\omega|$ (as compared with perturbations in a spheroidal field) does not exceed $1°$. Perturbations of the remaining parameters in this case are limited to the following values: $|\delta p| \leq 300$ m; $|\delta e| < 0.7 \cdot 10^{-4}$; $|\delta t| < 0^s.25$; $|\delta r| \leq 600$ m. Quasisecular disturbances in this case do not exceed the quantities: $|\delta p| \leq 200$ m; $|\delta e| \leq 0.1 \cdot 10^{-3}$; $|\delta\omega| \leq 3'.6$; $|\delta t| \leq 2^s.5$; $|\delta r| \leq 750$ m; $|\delta\Omega| \leq 3''$; $|\delta i| \leq 1'$. /144

"...It is frequently very rigorously proved that a solution for some problem exists, and it is theoretically established that this solution may be found with any degree of accuracy, and much less attention is paid to the critical part of the matter, i.e. to actually finding the solution". /145

"...It is not the computational process, but the result, which is usually of interest in applications; it is for this reason that an attempt is made to obtain this result with sufficient accuracy and with the minimum expenditure of time and labor".

A. N. Krylov

Chapter Three

APPROXIMATE METHODS FOR DESCRIBING DISTURBED MOTION OF AN ARTIFICIAL EARTH SATELLITE

The system of differential equations for disturbed motion (0.2) cannot be integrated in closed form. However, there are various methods for finding approximate solutions which describe the motion of a satellite in an eccentric gravitational field. Quite a few of these approximate methods are presently known. However, some of them, although they may be used for analytically studying the nature of disturbances and therefore have a certain practical value, have not come into extensive practical use because of the limitations placed on the possible values of the initial parameters. These include methods such as that described by V. F. Proskurin and Yu. V. Batrakov [52,53]. Considering the gravitational potential of a spheroid (with regard to the first power of flattening alone), they used a well-known method of celestial mechanics to expand the disturbing function in a series with respect to powers of eccentricity (the coefficients in this case are functions of the initial inclination) and obtained solutions of the Lagrange equations [1, 2] with respect to parameters a, e, i, Ω, $\pi = \Omega + \omega_3$ and ε (ε is the mean longitude in the period;(see [1, 2, 3]) in the form of sums of trigonometric functions which are multiples of the mean anomaly M. These solutions cannot be used for low values of eccentricity and circular orbits because of the selected system of parameters, nor for high values of eccentricity because of the limited range of convergence of the expansions (in which terms to e^4 /146 inclusive are retained).

The method of representing disturbed motion developed in the United States by J. Kozai [54, 55] is also inapplicable to nearly circular and circular satellites and in the neighborhood of the so-called "critical inclination" $i_0 = 63.4°$.[1] This last restiiction also applies to two other algorithms of non-Soviet authors [56, 59]. Limitations with respect to eccentricity and inclination impose considerable restrictions on the use of these methods when we recall that circular and nearly circular orbits are extremely desirable for many research purposes and for a number of technical problems, and the value $i_0 = 63.4°$ lies within the range of inclinations frequently used in practice (remember, the orbital planes of many Soviet satellites and space vehicles have roughly this same inclination).

This chapter brings together several algorithms which are free of the given limitations. They reflect various approaches to solution of the problem and may be used for direct calculations of disturbed satellite motion. Presentation does not necessarily include detailed exposition of the method of deriving the computational formulas, repetition of the calculations required, etc.

In those cases where all this is already available in works published by the authors, the ensuing sections pursue only the main goal--exposition of the algorithms in a form suitable for immediate practical application; here /147 we have limited ourselves merely to a brief description of the method which the author has created. The feasibility and convenience of direct practical utilization were the principal criteria for selecting the algorithms outlined below from among many others[2].

Since each of the given methods has its own advantages and disadvantages and may obviously have optimum application only for certain definite purposes, the last section contains a brief comparative analysis based chiefly on the results of calculations carried out by digital computer.

---

[1] The specific nature of this inclination boils down to the fact that there are no secular terms in the motion of the line of apsides for orbits with $i_0 = 63.4°$. The difficulties involved in describing motion in these cases (just as in the case of low eccentricities) are due only to the form of the description and have nothing whatsoever to do with any defects in the motion itself.

[2] As a specific instance, we do not take up the method developed by L. M. Lakhtin [60] even though it gives a fairly graphic geometric description of satellite motion.

§11. Use of the Small Parameter Method For Solving Equations of Disturbed Satellite Motion

In cases where satellite motion is being considered over a comparatively short time interval (about 24 hours), with the perigee of its orbit located at an altitude of about 300 km and its apogee at an altitude of less than six Earth radii, and the allowable error in position (coordinates) is a few km, first-order polar oblateness of the Earth is the only disturbing factor which needs to be considered on the basis of the results of the first and second chapters. In other words, model Б may be taken as the model of the gravitational field (§5), disregarding the effect of all remaining factors (§1).

In this case, there are a number of advantages to the description of motion in terms of osculating elements. One of these advantages lies in the fact that these elements change extremely slowly (which consequently makes it possible to use asymptotic methods), and the right-hand members of the differential equations are a linear function of a small parameter (which, as we shall see later, facilitates use of the Poincaré method). Besides, since the remaining disturbing effects (see §1) enter additively into the right-hand members of the differential equations and each of the terms will also contain a small parameter, it is then possible in principle to find solutions which account for the corresponding disturbances. Within the framework of the first approximation (linear theory) of the Poincaré method considered in this section, they need simply be added to the element perturbations derived below[1]. /148

The system of equation in osculating elements for the case of motion in the gravitational field of a spheroid (model Б) may be given as follows:

$$\left.\begin{aligned}
&\frac{d\Omega}{du}=\varepsilon p_0\frac{1}{p^2}R\cos i\sin^2 u;\\
&\frac{d}{du}\log\tan i=\varepsilon p_0^2\frac{1}{p^2}R\sin u\cos u;\\
&\frac{dp^2}{du}=4\varepsilon p_0^2 R\sin^2 i\sin u\cos u;\\
&\frac{dq}{du}=\frac{1}{2}\varepsilon\frac{p_0^2}{p^2}R[2k\cos^2 i\sin^2 u+(q+\cos u+\\
&\quad+R\cos u)\sin^2 i\sin 2u-R(3\sin^2 i\sin^2 u-1)\sin u];\\
&\frac{dk}{du}=\frac{1}{2}\varepsilon\frac{p_0^2}{p^2}R[-2q\cos^2 i\sin^2 u+(k+\sin u+
\end{aligned}\right\}\qquad(11.1)$$

[1] Equations are given in Appendix IX which describe motion in the field of the spheroidal Earth with additional forces acting on the satellite.

$$+ R\sin u)\sin^2 i\sin 2u + R(3\sin^2 i\sin^2 u - 1)\cos u];$$

$$\left.\begin{aligned} &\frac{dt}{du} = \frac{p^{3/2}}{\sqrt{\mu}} R^{-2} + \varepsilon \frac{p_0^2}{\sqrt{\mu}} p^{-1/2} R^{-1}\cos^2 i\sin^2 u, \\ &R = 1 + q\cos u + k\sin u. \end{aligned}\right\} \quad (11.1)^*$$

Here we use the notation $\varepsilon = 3c_{20} r_0^2/p_0^2$. The parameter $|\varepsilon|$ decreases with an increase in the geometric dimensions of the orbit, i.e. in the focal parameter p_0, the parameter $|\varepsilon|$ being small enough (when I. D. Zhongolovich's values are used for the coefficients in the expansion for the gravitational potential, $|\varepsilon| < 3\cdot 0.108586\cdot 10^{-2}$), so that asymptotic methods may be used for solving system (11.1). In the given case, when the solution is being sought over a comparatively small range of variation in the argument, it is better not to use the method of averaging (see next section), but rather the method of expanding the solution in series which are arranged in powers of the small parameter and are a special case of asymptotic Poincaré series[1]. /149

The principal disadvantage of this method is the appearance of secular terms of the form $u^n \sin^m u$, $u^i \cos^j u$ $(n, m, i, j = 1, 2....)$. It is not always a simple matter to determine whether they are a consequence of using the method or stem from the physical essence of the problem (in the case considered below, they appear during calculation of the second approximation).

This situation limits the range (the interval of variation in the argument) of application for solutions found by using asymptotic series. However, the method gives fairly accurate results for times of motion of about 24 hours[2].

By using the components of the Laplace vector in (11.1) as the parameters characterizing the form of the orbit and its position in the plane, we allow for examination of the entire range of eccentricities, while writing the second and third equations with respect to log tan i and p^2 (rather than

[1] We are not speaking of the Poincare method of finding periodic solutions, (see for instance [6], but only of using asymptotic Poincare series.
[2] The method of using series considered in this section should be distinguished from the expansions in powers of orbital eccentricity which are widely used in celestial mechanics (see for instance [2]), as the latter are true only for a certain range of initial parameters determined by the radius of convergence.

with respect to i and p) increases the accuracy of determining the unknown functions in the given case. And so, for solving system (11.1), we represent the unknown functions in the form of series in powers of the small parameter ($|\varepsilon| \ll 1$):

$$\left.\begin{aligned} \Omega &= \Omega_0 + \varepsilon F_{\Omega_1} + \varepsilon^2 F_{\Omega_2} + \dots; \\ \log \tan i &= \log \tan g_{i_0} + \varepsilon F_{i_1} + \varepsilon^2 F_{i_2} + \dots; \\ p^2 &= p_0^2(1 + \varepsilon F_{p_1} + \varepsilon^2 F_{p_2} + \dots); \\ q &= q_0 + \varepsilon F_{q_1} + \varepsilon^2 F_{q_2} + \dots; \\ k &= k_0 + \varepsilon F_{k_1} + \varepsilon^2 F_{k_2} + \dots . \end{aligned}\right\} \tag{11.2}$$

The last equation splits off from system (11.1), and for known functions p(u), q(u), k(u), i(u) is a quadrature which is subsequently computed inde- /150
pendently[1]. All functions F_{Ω_1}, $F_{\Omega_2}, \dots$, F_{i_1}, $F_{i_2}, \dots$, etc. depend on the initial parameters of motion and the argument u.

In the general case, the asymptotic property of series of form (11.2) lies in the fact that when $\varepsilon = 0$ they describe undisturbed motion of the object, and also, although generally speaking, similar series may diverge when $n \to \infty$ (i.e., with an unbounded increase in the number of terms of the series), they describe true motion with sufficient accuracy for fixed n and $\varepsilon \to 0$.

The first to use series of this type for solving systems of differential equations was H. Poincaré [62][2].

Let us follow this procedure. For the sake of simplicity, we write system (11.1) in the general form (without the last equation):

$$\left.\begin{aligned} d\Omega/du &= \Phi_\Omega(\Omega, i, p, q, k, u; \varepsilon); \\ d\ln \mathrm{tg}\, i/du &= \Phi_i(\Omega, i, p, q, k, u; \varepsilon); \\ dp^2/du &= \Phi_p(\Omega, i, p, q, k, u; \varepsilon); \\ dq/du &= \Phi_q(\Omega, i, p, q, k, u; \varepsilon); \\ dk/du &= \Phi_k(\Omega, i, p, q, k, u; \varepsilon). \end{aligned}\right\} \tag{11.3}$$

---

[1] When we wrote the system of motion equations (12.1) with respect to an angular argument rather than time, we reduced the order of the system by one.
[2] Actually, in this work Poincaré used asymptotic series for solving systems no higher than the second degree. V. V. Golubev gives a partial exposition of his method [63] (see also [2] on this point).

In addition to (11.3), let us consider the "undisturbed" (trivial in the given case) system which describes Keplerian motion:

$$\left.\begin{aligned} d\Omega_0/du &= 0; \\ d\log\tan i_0/du &= 0; \\ dp_0^2/du &= 0; \\ dq_0/du &= 0; \\ dk_0/du &= 0. \end{aligned}\right\} \qquad (11.4)$$

Let us expand the functions Φ_Ω, Φ_i, Φ_p etc. on the hypersurface $S^* = S^*(\Omega_0, i_0, p_0, q_0, k_0, u; \varepsilon = 0)$ in Taylor series with respect to the powers of small differences: $\Omega - \Omega_0$, $\log\tan i - \log\tan i_0$, $p^2 - p_0^2$, $q - q_0$, $k - k_0$, $\varepsilon - 0 = \varepsilon$. We get /151

$$\Phi(\Omega, i, p, q, k, u; \varepsilon) = \Phi_0(\Omega_0, i_0, p_0, q_0, k_0, u; \varepsilon = 0) + \frac{\partial\Phi_0}{\partial p}(p^2 - p_0^2) +$$

$$+ \frac{\partial\Phi_0}{\partial q}(q - q_0) + \frac{\partial\Phi_0}{\partial k}(k - k_0) + \frac{\partial\Phi_0}{\partial\Omega}(\Omega - \Omega_0) + \frac{\partial\Phi_0}{\partial\ln\operatorname{tg} i}(\ln\operatorname{tg} i - \ln\operatorname{tg} i_0) +$$

$$+ \frac{\partial\Phi_0}{\partial\varepsilon}\varepsilon + \frac{\partial^2\Phi_0}{\partial p^2}(p^2 - p_0^2)^2 + \ldots + \frac{\partial^2\Phi_0}{\partial\varepsilon\partial p}\varepsilon(p - p_0) + \frac{\partial^2\Phi_0}{\partial\varepsilon\partial q}\varepsilon(q - q_0) + \ldots$$

(the subscripts p, q, k, etc. for function Φ are omitted). Since the expansions are done on the surface S^*, where $\varepsilon = 0$, and since the parameter ε appears as a factor in the right-hand members of (11.3), the expansion of the functions Φ_Ω, Φ_i, Φ_p etc. retain only those terms with simple or mixed partial derivatives with respect to ε (i.e. $\partial\Phi_\Omega/\partial_\varepsilon$, $\partial\Phi_i/\partial\varepsilon, \ldots$, $\partial^2\Phi_\Omega/\partial\varepsilon\partial i$, $\partial^2\Phi_\Omega/\partial\varepsilon\partial p, \ldots$, etc.). Thus, functions Φ will take the form

$$\Phi(\Omega, i, p, q, k, u; \varepsilon) = \left(\frac{\partial\Phi}{\partial\varepsilon}\right)_0 \varepsilon + \left(\frac{\partial^2\Phi}{\partial\varepsilon\partial\Omega}\right)_0 \varepsilon(\Omega - \Omega_0) +$$

$$+ \left(\frac{\partial^2\Phi}{\partial\varepsilon\partial i}\right)_0 \varepsilon(\ln\operatorname{tg} i - \ln\operatorname{tg} i_0) + \left(\frac{\partial^2\Phi}{\partial\varepsilon\partial p}\right)_0 \varepsilon(p^2 - p_0^2) +$$

$$+ \left(\frac{\partial^2\Phi}{\partial\varepsilon\partial q}\right)_0 \varepsilon(q - q_0) + \left(\frac{\partial^2\Phi}{\partial\varepsilon\partial k}\right)_0 \varepsilon(k - k_0) + \ldots . \qquad (11.5)$$

The subscript 0 indicates that they are considered on hypersurface S^*.

We use series (11.2) to eliminate the differences in parentheses; we then differentiate (11.2) with respect to argument u and eliminate derivatives $d\Omega/du$, $d/du \log\tan u$, etc. from the left-hand members of (11.3). If we now substitute expansions (11.5) in the right-hand members and subtract the corresponding equations of system (11.4) from the equations in the resultant system, then by adding the coefficients associated with identical powers of ε in the right-hand and left-hand members of each equation, we get an infinite system of recurrent differential equations for determining the functions F_{Ω_1}, $F_{\Omega_2}, \ldots$, F_{i_1}, $F_{i_2}, \ldots$, etc. By using these equations, functions with a higher index may be determined in terms of functions with a lower index. In this way, all coefficients associated with powers of ε in (11.2) may be determined.

According to Poincare's theorem, assuming fulfillment of the conditions for continuity in the right-hand members of equations (11.3) with respect to the argument (in the given case with respect to u) and analyticity of these equations with respect to small absolute values of ε, $\Omega - \Omega_0$, log tan i - log tan i_0, $p^2 - p_0^2$, $q - q_0$ and $k - k_0$, series (11.2) will converge (in the usual sense) for sufficiently small values of $|\varepsilon|$ and will be a solution of the Cauchy problem for system of equations (11.3). This requirement is fulfilled in the case we are considering. /152

By fixing a definite number of terms in series (11.2), we will solve the problem with a certain degree of accuracy which naturally depends on the quantity $|\varepsilon|$. We may speak of approximations (first, second, etc.) as related to the terms to which they are limited in the expansions (the first power of ε, the second, etc.). Approximations higher than the second are not used in practice.

The fact that the parameter ε appears as a factor in the right-hand members of the equations appreciably simplifies finding the first and second approximations. The computational work boils down to solving integrals of the form

$$\int_{u_0}^{u} u^j \sin^m u \cos^n u\, du,\quad j=0,1,2,\ldots,\ m=0,1,2,\ldots,\ n=0,1,2,\ldots . \tag{11.6}$$

By applying the described procedure to system (11.1), we get the following first-approximation equations for determining the functions F:

$$\left.\begin{aligned}
\frac{dF_\Omega}{du} &= \left(\frac{\partial \Phi_\Omega}{\partial \varepsilon}\right)_0 = \beta[\sin^2 u + k_0 \sin^3 u + q_0 \sin^2 u \cos u];\\
\frac{dF_i}{du} &= \left(\frac{\partial \Phi_i}{\partial \varepsilon}\right)_0 = \sin u \cos u - q_0 \cos^2 u \sin u + k_0 \sin^2 u \cos u;\\
\frac{dF_p}{du} &= \left(\frac{\partial \Phi_p}{\partial \varepsilon}\right)_0 = 4p_0^2 \alpha^2 (\sin u \cos u + q_0 \sin u \cos^2 u + k_0 \sin^2 u \cos u);
\end{aligned}\right\} \tag{11.7}$$

$$\left.\begin{aligned}
\frac{dF_q}{du} &= \left(\frac{\partial\Phi_q}{\partial\varepsilon}\right)_0 = \frac{1}{2}R_0[\sin u+(1+2\beta^2)k_0\sin^2u-3\alpha^2\sin^3u-\\
&-3k_0\alpha^2\sin^4u+(1+2\alpha^2)q_0\sin u\cos u+4\alpha^2\cos^2u\sin u+\\
&+2k_0\alpha^2\sin^2u\cos^3u-3q_0\alpha^2\sin^3u\cos u+2q_0\alpha^2\cos^3u\sin u];\\
\frac{dF_k}{du} &= \left(\frac{\partial\Phi_k}{\partial\varepsilon}\right)_0 = \frac{1}{2}R_0[-\cos u-q_0\cos^2u-2q_0\beta^2\sin^2u+\\
&+(2\alpha^2-1)k_0\sin u\cos u+7\alpha^2\sin^2u\cos u+\\
&+5\alpha^2q_0\sin^2u\cos^2u+5\alpha^2k_0\sin^3u\cos u].
\end{aligned}\right\} \qquad (11.7)$$

Here, we use the notation /153

$$R_0 = 1+q_0\cos u+k_0\sin u;$$
$$\alpha = \sin i_0;$$
$$\beta = \cos i_0.$$

As we see, the system has been broken down into five independent equations with solutions which reduced to integrating the right-hand members with respect to u.

For computing the functions of the second approximations, we get the following recurrent formulas:

$$\left.\begin{aligned}
\frac{dF_{\Omega_2}}{du} &= \left[\left(\frac{\partial\Phi_\Omega}{\partial\varepsilon\partial q}\right)_0 F_{q_1}+\left(\frac{\partial\Phi_\Omega}{\partial\varepsilon\partial k}\right)_0 F_{k_1}+\left(\frac{\partial\Phi_\Omega}{\partial\varepsilon\partial i}\right)_0 F_{i_1}\right]=\\
&=\frac{1}{2}\beta[F_{q_1}\sin^2u\cos u+F_{k_1}\sin^3u]-\frac{1}{2}\alpha R_0F_{i_1}\sin^2u;\\
\frac{dF_{i_2}}{du} &= \left(\frac{\partial\Phi_i}{\partial\varepsilon\partial q}\right)_0 F_{q_1}+\left(\frac{\partial\Phi_i}{\partial\varepsilon\partial k}\right)_0 F_{k_1}=\\
&=\frac{1}{2}F_{q_1}\cos^2u\sin u+\frac{1}{2}F_{k_1}\sin^2u\cos u;\\
\frac{dF_{p_2}}{du} &= \left(\frac{\partial\Phi_p}{\partial\varepsilon\partial i}\right)_0 F_{i_1}+\left(\frac{\partial\Phi_p}{\partial\varepsilon\partial q}\right)_0 F_{q_1}+\left(\frac{\partial\Phi_p}{\partial\varepsilon\partial k}\right)_0 F_{k_1}=\\
&=2p_0^2[\alpha^2F_{q_1}\cos^2u\sin u+\alpha^2F_{k_1}\sin^2u\cos u+\\
&\qquad+2\alpha\beta R_0F_{i_1}\sin u\cos u];
\end{aligned}\right\} \qquad (11.8)$$

$$\left.\begin{aligned}
\frac{dF_{q_2}}{du} &= \left(\frac{\partial\Phi_q}{\partial\varepsilon\partial i}\right)_0 F_{i_1} + \left(\frac{\partial\Phi_q}{\partial\varepsilon\partial p}\right)_0 F_{p_1} + \left(\frac{\partial\Phi_q}{\partial\varepsilon\partial q}\right)_0 F_{q_1} + \\
&\quad + \left(\frac{\partial\Phi_q}{\partial\varepsilon\partial k}\right)_0 F_{k_1};\\
\frac{dF_{k_2}}{du} &= \left(\frac{\partial\Phi_k}{\partial\varepsilon\partial i}\right)_0 F_{i_1} + \left(\frac{\partial\Phi_k}{\partial\varepsilon\partial p}\right)_0 F_{p_1} + \left(\frac{\partial\Phi_k}{\partial\varepsilon\partial q}\right)_0 F_{q_1} + \\
&\quad + \left(\frac{\partial\Phi_k}{\partial\varepsilon\partial k}\right)_0 F_{k_1}.
\end{aligned}\right\} \qquad (11.8)$$

The first approximation solutions (expansions with accuracy to ε) show a fair degree of accuracy after checking. Therefore, there is no need for introducing the second approximation in its entirety into the final formulas. It is sufficient to include in these formulas only the most significant terms from the second approximation, in particular those which appear as quasisecular terms in the equations for q, k and Ω. The advisability of writing the second and third equations of system (11.3) with respect to log tan i and p^2 now becomes obvious from an examination of the second approximation functions /154

$$\frac{dF_{i_2}}{du} \quad \text{and} \quad \frac{dF_{p_2}}{du}.$$

In the case where the equations are written in form (11.1), they are equal to

$$\frac{dF_{i_2}}{du} = \frac{1}{2}(F_{q_1}\cos^2 u \sin u + F_{k_1}\sin^2 u \cos u);$$

$$\frac{dF_{p_2}}{du} = 2p_0^2\,[F_{q_1}\sin^2 i_0 \cos^2 u \sin u + F_{k_1}\sin^2 i_0 \sin^2 u \cos u + {}$$
$$+ 2F_{i_1}(1 + q_0\cos u + k_0 \sin u)\sin i_0 \cos i_0 \sin u \cos u].$$

If the same equations are written in the usual form $\frac{di}{du} = \ldots;$ $\frac{dp}{du} = \ldots$, then we get in the second approximation

$$\frac{dF_{i_2}}{du} = \frac{1}{2}\,[F_{q_1}\sin i_0\cos i_0\cos^2 u\sin u + F_{k_1}\sin i_0\cos i_0\sin^2 u\cos u + F_{i_1}(1+q_0\cos u + k_0\sin u)\cos^2 i_0\sin u\cos u];$$

$$\frac{dF_{p_2}}{du} = 2p_0^2\,[F_{q_1}\sin^2 i_0\cos^2 u\sin u + F_{k_1}\sin^2 i_0\sin^2 u\cos u + 2F_{i_1}(1+q_0\cos u + k_0\sin u)\sin i_0\cos i_0\sin u\cos u - \frac{1}{p_0^2}F_{p_1}(1+q_0\cos u + k_0\sin u)\sin^2 i_0\sin u\cos u].$$

As may be seen, the form of the second-approximation functions in the first case is considerably simpler, i.e. the greatest number of harmonics representing motion went into the first approximation. This is an indication that the first approximations already represent the unknown functions fairly well, thus providing an accurate solution. The final solutions for the five equations of system (11.1) are as follows:

/155

$$\left.\begin{aligned}
\Omega &= \Omega_0 + \varepsilon\beta\Big[C_0\,\Delta(u) + \tfrac{1}{3}q_0\,\Delta(\sin^3 u) - k_0\,\Delta(\cos u) + \tfrac{1}{3}k_0\,\Delta(\cos^3 u) - \tfrac{1}{4}\Delta(\sin 2u)\Big];\\
i &= \operatorname{arctg}\Big[\operatorname{tg} i_0\exp\Big\{\varepsilon\Big[\tfrac{1}{2}\Delta(\sin^2 u) + \tfrac{1}{3}k_0\,\Delta(\sin^3 u) - \tfrac{1}{3}q_0\,\Delta(\cos^3 u)\Big]\Big\}\Big];
\end{aligned}\right\} \quad (11.9)$$

$$\left.\begin{aligned}
p &= p_0\Big\{1 + 4\varepsilon\alpha^2\Big[\tfrac{1}{2}\Delta(\sin^2 u) + \tfrac{1}{3}k_0\,\Delta(\sin^3 u) - \tfrac{1}{3}q_0\,\Delta(\cos^3 u)\Big]\Big\}^{1/2};\\
q &= q_0 + \varepsilon[A_0\,\Delta(u) + A_1\,\Delta(\sin u) + A_2\,\Delta(\sin^2 u) + A_3\,\Delta(\sin^3 u) + A_4\,\Delta(\sin^4 u) + A_5\,\Delta(\sin^5 u) +\\
&\quad + A_6\,\Delta(\cos u) + A_7\,\Delta(\cos^3 u) + A_8\,\Delta(\cos^4 u) + A_9\,\Delta(\cos^5 u) + A_{10}\,\Delta(\cos^3 u\sin u) + A_{11}\,\Delta(\cos^4 u\sin u) +\\
&\quad + A_{12}\,\Delta(\sin^4 u\cos u) + A_{13}\,\Delta(\sin 2u)];\\
k &= k_0 + \varepsilon[B_0\,\Delta(u) + B_1\,\Delta(\sin u) + B_2\,\Delta(\sin^2 u) + B_3\,\Delta(\sin^3 u) + B_4\,\Delta(\sin^4 u) + B_5\,\Delta(\sin^5 u) + B_6\,\Delta(\cos u) +\\
&\quad + B_7\,\Delta(\cos^3 u) + B_8\,\Delta(\sin 2u) + B_9\,\Delta(\cos^3 u\sin u) + B_{10}\,\Delta(\cos^4 u\sin u) + B_{11}\,\Delta(\sin^4 u\cos u)].
\end{aligned}\right\}$$

Here, we use the notation $\Delta(u) = u - u_0$; $\Delta(\sin u) = \sin u = \sin u_0$, etc.

$$C_0 = \frac{1}{2} - 0{,}221\varepsilon;$$

$$A_0 = k_0\left(\beta^2 - \frac{1}{4}\alpha^2\right) + \varepsilon Q_0;$$

$$A_1 = \frac{2}{5} q_0 k_0 \alpha^2;$$

$$A_2 = \frac{1}{2} q_0 (1 + \alpha^2);$$

$$A_3 = \frac{2}{3}\left(1 - \frac{1}{5}\alpha^2\right) q_0 k_0;$$

$$A_4 = -\frac{3}{4} q_0 \alpha^2;$$

$$A_5 = -\frac{3}{5} q_0 k_0 \alpha^2;$$

$$A_6 = \frac{3}{2}\alpha^2 - \frac{1}{2} - \frac{3}{2}\left(k_0^2 + \frac{1}{5} q_0^2 \alpha^2\right);$$

$$A_7 = -\frac{7}{6}\alpha^2 - \frac{1}{3}\left(\frac{1}{2} + \frac{13}{10}\alpha^2\right) q_0^2 + \frac{1}{3}\left(\frac{1}{2} + \beta^2 - \frac{6}{5}\alpha^2\right) k_0^2;$$

$$A_8 = -\frac{3}{4} q_0 \alpha^2;$$

$$A_9 = -\frac{1}{5} q_0^2 \alpha^2;$$

$$A_{10} = -\frac{3}{2} k_0 \alpha^2;$$

$$A_{11} = -\frac{2}{5} q_0 k_0 \alpha^2;$$ /156

$$A_{12} = \alpha^2\left(\frac{1}{2} k_0^2 - \frac{3}{10} q_0^2\right);$$

$$A_{13} = k_0\left(\frac{7}{8}\alpha^2 - \frac{1}{2}\beta^2\right);$$

$$B_0 = q_0\left(\frac{1}{4}\alpha^2 - \beta^2\right) + \varepsilon K_0;$$

$$B_1 = -\frac{1}{2} q_0^2 \beta^2;$$

$$B_2 = -\frac{1}{2} k_0 \beta^2;$$

$$B_3 = \frac{1}{3}\alpha^2\left(k_0^2 - \frac{1}{2} q_0^2\right) - \frac{1}{6}(k_0^2 - q_0^2) - \frac{1}{3} q_0^2 \beta^2 + \frac{7}{6}\alpha^2;$$

$$B_4 = \frac{3}{2} k_0 \alpha^2;$$

$$B_5 = \frac{1}{2} k_0^2 \alpha^2;$$

$$B_6 = q_0 k_0 (\beta^2 - \alpha^2);$$

$$B_7 = \frac{1}{3} q_0 k_0 \alpha^2;$$

$$B_8 = \frac{1}{8} q_0 \alpha^2;$$

$$B_9 = -\frac{3}{2} q_0 \alpha^2;$$

$$B_{10} = -\frac{1}{2} q_0^2 \alpha^2;$$

$$B_{11} = q_0 k_0 \alpha^2;$$

$$Q_0 = 1.125\alpha^4 - 0.5\alpha^3 - 0.5\alpha^2 + 0.125\alpha\beta^2;$$

$$K_0 = -0.375 + 2.187176\alpha - 1.7704869\alpha^2 - \\ - 1.6062218\alpha\beta^2 - 0.1878664\beta^2.$$

The five equations obtained in form (11.9) describe the evolution of the osculating orbit of an artificial satellite. Description of satellite motion requires still another equation which, specifically, may be the relationship t(u).

This function, as has already been pointed out previously, is found by solving the last equation in system (11.1),

$$\frac{dt}{du} = \frac{p^{3/2}}{\sqrt{\mu}} R^{-2} + \varepsilon \frac{p_0^2}{\sqrt{\mu}} p^{-1/2} R^{-1} \cos^2 i \sin^2 u$$

assuming that the functions p(u), i(u), q(u) and k(u) are known.

Thus, the problem reduces to computing the quadratures of /157

$$t = t_0 + \frac{1}{\sqrt{\mu}} \left[\int_{u_0}^{u} \frac{p^{3/2}}{(1+q\cos u + k\sin u)^2} du + \right. \\ \left. + \varepsilon p_0^2 \int_{u_0}^{u} \frac{\cos^2 i \sin^2 u}{\sqrt{p}(1+q\cos u + k\sin u)} du \right], \quad (11.10)$$

in which p(u), i(u), q(u) and K(u) are given by equations (11.9).

In the case where problems involving satellite motion are solved by digital computer, numerical quadrature of expressions (11.10) with the necessary accuracy should not be difficult. However, this does not obviate the search for other methods of determining the position of an artificial Earth satellite within the framework of the theory outlined in this section.

§12. Solving Equations of Disturbed Satellite Motion by the Averaging Method

This section outlines an approximate solution derived by Yu. G. Yevtushenko for calculating the evolution of satellite motion in the field of a triaxial asymmetric ellipsoid. Equations of disturbed satellite motion are integrated by the averaging method in a form developed by V. M. Volosov for systems with a rapidly rotating phase [64-67]. The feasibility of this approach to solving various problems in satellite dynamics was pointed out by N. N. Moiseyev [68]. Appendix X contains a brief exposition of the method.

The averaging method was used for studying satellite motion in the work of D. Ye. Okhotsimskiy, T. M. Eneyev and G. P. Taratynova [13] in a first-approximation investigation of the effect which eccentricity of the Earth's gravitational field has on perturbation of the satellite's orbital elements. Kozai [54] used the averaging method for a second-approximation study of the same problem. However, the solution found by Kozai does not give a complete description of the motion since the given solution may be used for examining only orbital evolution in the second approximation, while the position of the satellite is determined in the first approximation; besides this, the approximate solution cannot be used in the case of low initial eccentricities or when the inclination is close to 63.4°. The effect which equatorial flattening of the Earth has on satellite motion was not considered. /158

The asymptotic solution constructed below gives a second-approximation description of satellite motion in the field of a triaxial asymmetric ellipsoid. The approximate solution is valid for orbits with arbitrary inclinations and eccentricities less than unity. Special consideration is given to the case of nearly circular orbits.

This section will deal chiefly with derivation of the approximate solution. Qualitative analysis based on the resultant formulas is taken up in Appendices VII and VIII.

We shall use the symbol Λ to designate the geographic longitude of the satellite reckoned in the plane of the equator eastward from one of the semimajor axes of the equatorial ellipse.

From (6.6') we get the following expression for Λ:

$$\Lambda = \Omega - \omega_3 t + \arccos\frac{\cos u}{\sqrt{1-\sin^2 u \sin^2 i}} . \qquad (12.1)$$

Here, ω_3 is the angular rate of rotation of the Earth around the polar axis.

We shall write the potential for a triaxial asymmetric ellipsoid (model E, see §5) in the form

$$V = \frac{\mu}{r}\left[1 + c_{20}\left(\frac{r_0}{r}\right)^2 P_{20}(\sin\varphi) + c_{30}\left(\frac{r_0}{r}\right)^3 P_{30}(\sin\varphi) + c_{40}\left(\frac{r_0}{r}\right)^4 P_{40}(\sin\varphi) + \right.$$
$$\left. + \left(\frac{r_0}{r}\right)^2 \sqrt{c_{22}^2 + d_{22}^2}\, P_{22}(\sin\varphi)\cos 2\Lambda\right]. \tag{12.2}$$

The force due to eccentricity of the Earth's gravitational field is considerably less than the principal force of Newtonian attraction. Therefore, we take as the small parameter the quantity

$$\varkappa = -\varepsilon = -3c_{20} r_0^2 / p_0^2, \tag{12.3}$$

which is proportional to the ratio between acceleration due to first-order polar oblateness and acceleration due to gravity at altitude p_0.

The coefficients c_{30}, c_{40}, d_{22} and c_{22} in gravitational potential (12.2) are of the second negative order of magnitude with respect to polar flattening, and therefore, we assume

$$c_{40} = \frac{2c\varkappa^2 p_0^4}{5r_0^4}; \quad c_{30} = \frac{f\varkappa^2 p_0^3}{r_0^3}; \quad \sqrt{c_{22}^2 + d_{22}^2} = \frac{b\varkappa^2 p_0^2}{6r_0^2},$$

where c, f and b are some constants; c characterizes second polar flattening; /159
the coefficient b is due to equatorial flattening of the Earth.

Let us introduce the following dimensionless quantities into our analysis: time τ, focal parameter $\tilde{p}$, focal radius $\tilde{r}$ and average angular velocity of orbital motion $\tilde{n}$:

$$\tau = t\sqrt{\mu}\, p_0^{-3/2}; \quad \tilde{p} = pp_0^{-1}; \quad \tilde{r} = rp_0^{-1}; \quad \tilde{n} = (1-e^2)^{3/2}\,\tilde{p}^{-3/2}. \tag{12.3'}$$

The potential of the disturbing forces reduced to $\varkappa\mu p_0^{-1}$ we write in the form

$$U = \frac{p_0 V}{\varkappa\mu} = \frac{1}{6\tilde{r}^3}(1 - 3\sin^2\varphi) + \frac{\varkappa f}{2\tilde{r}^4}(5\sin^2\varphi - 3)\sin\varphi +$$
$$+ \frac{\varkappa c}{20\tilde{r}^5}[35\sin^4\varphi - 30\sin^2\varphi + 3] + \frac{\varkappa b}{2\tilde{r}^3}\cos 2\Lambda\cos^2\varphi. \tag{12.4}$$

When (12.4) is taken into consideration, the system reduces to the form

$$\left.\begin{aligned}
&\frac{d\Omega}{d\tau}=\frac{\varkappa\sqrt{\tilde{p}}\,\overline{W}\sin u}{R\sin i};\quad \frac{di}{d\tau}=\frac{\varkappa\sqrt{\tilde{p}}\,\overline{W}\cos u}{R};\quad \frac{d\tilde{p}}{d\tau}=\frac{2\varkappa\tilde{p}^{3/2}\,\overline{T}}{R};\\
&\frac{dq}{d\tau}=\varkappa\sqrt{\tilde{p}}\{\overline{S}\sin u+\overline{T}[\cos u+R^{-1}(q+\cos u)]+k\overline{W}R^{-1}\sin u\operatorname{ctg} i\};\\
&\frac{dk}{d\tau}=\varkappa\sqrt{\tilde{p}}\{-\overline{S}\cos u+\overline{T}[\sin u+R^{-1}(k+\sin u)]-q\overline{W}R^{-1}\sin u\operatorname{ctg} i\};\\
&\frac{du}{d\tau}=\frac{R^2}{\tilde{p}^{3/2}}-\frac{\varkappa\sqrt{\tilde{p}}}{R}\overline{W}\operatorname{ctg} i\sin u.
\end{aligned}\right\} \qquad (12.5)$$

Here $\overline{S}$, $\overline{T}$ and $\overline{W}$ are dimensionless components reduced to κ of the disturbing acceleration in the radial, transversal and normal directions to the plane of motion,respectively. The formulas for computing $\overline{S}$, $\overline{T}$ and $\overline{W}$ are analogous to formulas (6.4'):

$$\left.\begin{aligned}
&\overline{S}=\frac{\partial U}{\partial\tilde{r}};\quad \overline{T}=\frac{1}{\tilde{r}}\frac{\partial U}{\partial\varphi}\cdot\frac{\cos u}{\cos\varphi}\sin i+\frac{\cos i}{\tilde{r}\cos^2\varphi}\frac{\partial U}{\partial\Lambda};\\
&\overline{W}=\frac{1}{\tilde{r}}\frac{\cos i}{\cos\varphi}\frac{\partial U}{\partial\varphi}-\frac{\cos u\sin i}{\tilde{r}\cos^2\varphi}\frac{\partial U}{\partial\Lambda}.
\end{aligned}\right\} \qquad (12.5')$$

In order to reduce system (12.5) to the standard form (see Appendix X), we introduce the new variable

$$L=2\operatorname{arctg}\sqrt{\frac{1-e}{1+e}}\,\frac{q\sin u-k\cos u}{e+q\cos u+k\sin u}-\frac{\sqrt{1-e^2}\,(q\sin u-k\cos u)}{1+q\cos u+k\sin u}+\operatorname{arctg}\frac{k}{q}. \qquad (12.6)$$

Formula (12.6) has no singularities in the case of nearly circular orbits. Actually, by using the formula /160

$$\tan\frac{\vartheta}{2}=\frac{\sin\vartheta}{1+\cos\vartheta},$$

we transform (12.6) to the form

$$L=2\operatorname{arctg}\sqrt{\frac{1-e}{1+e}}\operatorname{tg}\frac{1}{2}\left[u-\operatorname{arctg}\frac{k}{q}\right]-\frac{\sqrt{1-e^2}(q\sin u-k\cos u)}{1+q\cos u+k\sin u}+\operatorname{arctg}\frac{k}{q}. \qquad (12.6')$$

By expanding the right-hand member of (12.6') in a series with respect to e and dropping small terms of order O(e), we get

$$L = u.$$

Thus, on nearly circular orbits L coincides with the argument of latitude.

Differentiating (12.6) with respect to q and k, we get

$$\left.\begin{aligned} \frac{\partial L}{\partial q} &= \frac{\sqrt{1-e^2}}{e^2 R^2}\left[-q(1+R)(q\sin u - k\cos u) + k(1-e^2) - \frac{kR^2}{\sqrt{1-e^2}}\right]; \\ \frac{\partial L}{\partial k} &= \frac{\sqrt{1-e^2}}{e^2 R^2}\left[-k(1+R)(q\sin u - k\cos u) - q(1-e^2) + \frac{qR^2}{\sqrt{1-e^2}}\right]. \end{aligned}\right\} \qquad (12.7)$$

In the light of system (12.5), we differentiate (12.6) and after transformations we get an equation for the variable L:

$$\frac{dL}{d\tau} = \tilde{n} - \frac{2\sqrt{\tilde{p}(1-e^2)}}{R}\,\overline{S} + \left(q\frac{dk}{d\tau} - k\frac{dq}{d\tau}\right)(1+\sqrt{1-e^2})^{-1} - \cos i\sqrt{1-e^2}\,\frac{d\Omega}{d\tau}. \qquad (12.8)$$

The angle of average anomaly M is expressed in terms of the elements of motion by the formula [1, 2]

$$M = 2\operatorname{arctg}\sqrt{\frac{1-e}{1+e}}\ \operatorname{tg}\frac{\vartheta}{2} - \frac{e\sqrt{1-e^2}\sin\vartheta}{1+e\cos\vartheta},$$

where $\vartheta = u - \omega$ is the true anomaly.

Considering the potential of the disturbing forces a function of Ω, i, $\tilde{n}$, k, q and M, we write the system of motion equations in Lagrange's form:

$$\left.\begin{aligned} &\frac{d\Omega}{d\tau} = \frac{\varkappa\tilde{n}^{1/3}}{\sin i\sqrt{1-e^2}}\frac{\partial U}{\partial i}; \quad \frac{d\tilde{n}}{d\tau} = -3\varkappa\tilde{n}^{4/3}\frac{\partial U}{\partial M}; \\ &\frac{di}{d\tau} = \frac{\varkappa\tilde{n}^{1/3}\cos i}{\sin i\sqrt{1-e^2}}\left[q\frac{\partial U}{\partial k} - k\frac{\partial U}{\partial q}\right] - \frac{\varkappa\tilde{n}^{1/3}}{\sin i\sqrt{1-e^2}}\frac{\partial U}{\partial \Omega}; \end{aligned}\right\} \qquad (12.9)$$

$$\left.\begin{aligned}
\frac{dq}{d\tau} &= \varkappa\tilde{n}^{1/3}\left[\frac{q(1-e^2)}{e^2}\frac{\partial U}{\partial M}+\frac{k\operatorname{ctg} i}{\sqrt{1-e^2}}\frac{\partial U}{\partial i}-\sqrt{1-e^2}\cdot\frac{\partial U}{\partial k}\right];\\
\frac{dk}{d\tau} &= \varkappa\tilde{n}^{1/3}\left[\frac{k(1-e^2)}{e^2}\frac{\partial U}{\partial M}-\frac{q\operatorname{ctg} i}{\sqrt{1-e^2}}\frac{\partial U}{\partial i}+\sqrt{1-e^2}\frac{\partial U}{\partial q}\right];\\
\frac{dM}{d\tau} &= \tilde{n}-\varkappa\frac{1-e^2}{e^2}\tilde{n}^{1/3}\left[q\frac{\partial U}{\partial q}+k\frac{\partial U}{\partial k}\right]+3\varkappa\tilde{n}^{4/3}\frac{\partial U}{\partial\tilde{n}}.
\end{aligned}\right\} \tag{12.9}$$ /161

We shall not dwell on the derivation of equations (12.9) since the procedure is analogous to that presented in [1 or 2].

If we disregard equatorial flattening (i.e. we assume b = 0 in 12.4), the angle Ω and time of motion τ will not appear in the expression for the potential of the disturbing forces. Thus, Ω will be a cyclic variable, while the gravitational field is a conservative variable. In this case, system (12.9) has the following two integrals: the moment of momentum with respect to the polar axis

$$I_1 = \sqrt{p}\cos i$$

and the total energy of the satellite

$$I_2 = \frac{\tilde{n}^{2/3}}{2}+\varkappa U.$$

Actually, differentiating I_1 and I_2 in conformity with (12.9), we get

$$\left.\begin{aligned}
\frac{dI_1}{d\tau} &= \frac{\partial I_1}{\partial\tilde{n}}\frac{d\tilde{n}}{d\tau}+\frac{\partial I_1}{\partial i}\frac{di}{d\tau}+\frac{\partial I_1}{\partial q}\frac{dq}{d\tau}+\frac{\partial I_1}{\partial k}\frac{dk}{d\tau}=0;\\
\frac{dI_2}{d\tau} &= \frac{\tilde{n}^{-1/3}}{3}\frac{d\tilde{n}}{d\tau}+\varkappa\left[\frac{\partial U}{\partial\tilde{n}}\frac{d\tilde{n}}{d\tau}+\frac{\partial U}{\partial i}\frac{di}{d\tau}+\frac{\partial U}{\partial q}\frac{dq}{d\tau}+\right.\\
&\qquad\left.+\frac{\partial U}{\partial k}\frac{dk}{d\tau}+\frac{\partial U}{\partial M}\frac{dM}{d\tau}\right]=-\varkappa\tilde{n}\frac{\partial U}{\partial M}+\varkappa\tilde{n}\frac{\partial U}{\partial M}=0.
\end{aligned}\right\} \tag{12.10}$$

The disturbing terms in (12.4) due to equatorial flattening cause a slow variation in I_1 and I_2,

Let us introduce a new variable F by the relationship

$$\tilde{n}^{2/3}-\tilde{n}_0^{2/3}+2\varkappa(U-U_0)=2\varkappa F. \tag{12.11}$$

From (12.11) we find /162

$$\tilde{n} = [\tilde{n}_0^{2/3} + 2\varkappa(F + U_0 - U)]^{3/2}. \tag{12.12}$$

Expression (12.12) may be simplified if the right-hand member is expanded in a series with respect to powers of small parameter κ. Disregarding terms of the third negative order of magnitude (order $O(\kappa^3)$), we get

$$\tilde{n} = \tilde{n}_0 + 3\varkappa \tilde{n}_0^{1/3}(F + U_0 - U)\left[1 + \frac{\varkappa \tilde{n}_0^{-2/3}}{2}(F + U_0 - U)\right]. \tag{12.13}$$

With an error of $\sim O(\kappa^2)$ we find from (12.13)

$$\tilde{n} = \tilde{n}_0 + 3\varkappa \tilde{n}_0^{1/3}(F + U_0 - U). \tag{12.14}$$

By differentiating (12.11) in the light of (12.5), we find an equation for F:

$$dF/d\tau = \partial U/\partial \tau.$$

The equation for mean anomaly has a singularity on nearly circular orbits, therefore, we shall use the angle L in the future. Let us substitute expression (12.13) for $\tilde{n}$ in the right-hand member of equation (12.8). System of equations (12.5), (12.8) reduces to the standard form of systems with a rapidly rotating phase (see Appendix X):

$$\left.\begin{aligned}
&\frac{d\Omega}{d\tau} = \frac{\varkappa\sqrt{\tilde{p}}\,\overline{W}\sin u}{R\sin i}; \quad \frac{di}{d\tau} = \frac{\varkappa\sqrt{\tilde{p}}\,\overline{W}\cos u}{R};\\
&\frac{dq}{d\tau} = \varkappa\sqrt{\tilde{p}}\{\overline{S}\sin u + \overline{T}[(q + \cos u)R^{-1} + \cos u] +\\
&\qquad\qquad + kR^{-1}\overline{W}\sin u \operatorname{ctg} i\};\\
&\frac{d\tilde{p}}{d\tau} = \frac{2\varkappa\tilde{p}^{3/2}}{R}\overline{T}; \quad \frac{dF}{d\tau} = \frac{\partial U}{d\tau};\\
&\frac{dL}{d\tau} = \tilde{n}_0 - \frac{2\varkappa}{R}\sqrt{\tilde{p}(1-e^2)}\,\overline{S} - 3\varkappa n^{1/3}U + 3\varkappa \tilde{n}_0^{1/3}(F + U_0) +\\
&\qquad + \frac{3\varkappa^2}{2\tilde{n}^{1/3}}[(F + U_0)^2 - U^2] + \left(q\frac{dk}{d\tau} - k\frac{dq}{d\tau}\right)(1 + \sqrt{1-e^2})^{-1} -\\
&\qquad\qquad - \sqrt{1-e^2}\cos i\,\frac{d\Omega}{d\tau}.
\end{aligned}\right\} \tag{12.15}$$

The eccentricity which appears in the right-hand members of the equations in system (12.15) is expressed in terms of the orbital elements by formulas (12.3') and (12.13). Equations (12.15) are found with an error of order κ^3.

The right-hand members of the first five equations in system (12.15) are proportional to small parameter κ, and therefore the variables Ω, F, i, q, and $\tilde{p}$ change slowly. The varible L changes comparatively rapidly since /163

$$dL/d\tau \approx \tilde{n}_0 \gg \varkappa .$$

When $\kappa = 0$, system (12.15) describes undisturbed (Keplerian) motion of the satellite

$$L = \tilde{n}_0 \tau .$$

In this case, the variables Ω, F, i, q and $\tilde{p}$ are constant.

System (12.15) has no singularities at any eccentricities less than unity. Therefore, the method of asymptotic integration may be used in this system.

Let us reduce system (12.15) to a simpler averaged system in which the slowly changing variables and rapidly changing L will be separated. For this purpose we use the standard substitution of variables:

$$\left.\begin{aligned} \Omega &= \bar{\Omega} + \varkappa\Omega_1 + \varkappa^2\Omega_2 + \varkappa^3 \ldots, & i &= \bar{i} + \varkappa i_1 + \varkappa^2 \ldots ; \\ \tilde{p} &= \bar{p} + \varkappa p_1 + \varkappa^2 \ldots, & F &= \bar{\bar{F}} + \varkappa F_1 + \varkappa^2 \ldots ; \\ q &= \bar{q} + \varkappa q_1 + \varkappa^2 \ldots, & L &= \bar{L} + \varkappa L_1 + \varkappa^2 \ldots . \end{aligned}\right\} \quad (12.16)$$

Here, Ω_1, i_1, F_1, q_1, p_1, L_1 are some as yet unknown functions of the new variables $\bar{\Omega}$, $\bar{i}$, $\bar{F}$, $\bar{q}$, $\bar{p}$, $\bar{L}$. The variables $\bar{\Omega}$, $\bar{i}$, $\bar{F}$, $\bar{q}$, $\bar{p}$, $\bar{L}$ satisfy the averaged system

$$\left.\begin{aligned} \frac{d\bar{\Omega}}{d\tau} &= \varkappa A_{1\Omega} + \varkappa^2 A_{2\Omega} + \ldots, & \frac{d\bar{\bar{F}}}{d\tau} &= \varkappa A_{1F} + \varkappa^2 A_{2F} + \ldots ; \\ \frac{d\bar{i}}{d\tau} &= \varkappa A_{1i} + \varkappa^2 A_{2i} + \ldots, & \frac{d\bar{p}}{d\tau} &= \varkappa A_{1p} + \varkappa^2 A_{2p} + \ldots ; \\ \frac{d\bar{q}}{d\tau} &= \varkappa A_{1q} + \varkappa^2 A_{2q} + \ldots, & \frac{d\bar{L}}{d\tau} &= \tilde{n}_0 + \varkappa B_1 + \varkappa^2 B_2 + \ldots , \end{aligned}\right\} \quad (12.17)$$

where $A_{1\Omega}$, $A_{2\Omega}, \ldots$, B_1, B_2 are functions of the slowly changing variables $\overline{\Omega}$, $\overline{i}$, $\overline{F}$, $\overline{q}$, $\overline{p}$.

The physical meaning of transformation (12.16) lies in dissociation of the real motion described by the variables Ω, i, F, q, p and L into averaged (secular and log-period) motion $\overline{\Omega}$, $\overline{i}$, $\overline{F}$, $\overline{q}$, $\overline{p}$, $\overline{L}$ and short-period motion which is described by the functions

$$\Omega_1, i_1, F_1, q_1, p_1, L_1, \Omega_2, i_2, \ldots .$$

The algorithm for finding functions Ω_1, $i_1, \ldots$, $A_{1\Omega}$, $A_{2\Omega}, \ldots$, B_1, B_2 is given in Appendix X.

Averaged system (12.17) is found in the first approximation by averaging the right-hand members of the equations in system (12.15). The result of the averaging will depend on whether or not the periods of orbital motion of the satellite are commensurate with the period of the Earth's rotation. /164

We shall say that resonance takes place if the frequency of orbital motion of the satellite and the frequency of the Earth's rotation about the polar axis are commensurate, i.e. if there exist mutually simple numbers m and s such that

$$\tilde{n}_0 = \frac{m}{s} \overline{\omega}_3 .$$

Here $\overline{\omega}_3 = \omega_3 p_0^{3/2} \mu^{-1/2}$ is the dimensionless angular rotational velocity of the Earth.

Let us illustrate the singularity of computations of an averaged system through the example of finding the mean value of the function $\sin 2\overline{\omega}_3 \tau f(\overline{L}, \overline{\Omega}, \overline{i}, \overline{k}, \overline{q}, \overline{F})$ where $f(\overline{L}, \overline{\Omega}, \overline{i}, \overline{k}, \overline{q}, \overline{F})$ is some periodic function of $\overline{L}$ with period 2π. The mean value is computed by the formula

$$I_3' = \lim_{T \to \infty} \frac{1}{T} \int_0^T \sin 2\overline{\omega}_3 \tau f d\tau . \tag{12.18}$$

Integral (12.18) is computed along the trajectory of undisturbed motion, i.e. at constant values of the slowly changing variables. In this case,

$\overline{L} = n_0\tau$, therefore, the function f is periodic with respect to τ with period $2\pi n_0^{-1}$.

Let us expand the function $f(\tau)$ in a Fourier series with respect to $\cos n_0 j\tau$, $\sin n_0 j\tau$. We substitute the resultant series in (12.18), and the integrand becomes a sum of terms of the form

$$a_j \sin 2\overline{\omega}_3 \tau \cos n_0 j\tau + b_j \sin 2\overline{\omega}_3 \tau \sin n_0 j\tau . \tag{12.19}$$

Integrating (12.18) with respect to τ from $\tau = 0$ to $\tau = T$, we get a convergent series which consists of trigonometric functions of the variable τ and is therefore bounded as $T \to \infty$. Letting $T \to \infty$, we find that the mean value of the function $\sin 2\omega_3\tau f(\overline{L})$ is equal to zero.

In the resonance case, the integral (12.18) will differ from zero if

$$2\omega_3 = jn_0 , \tag{12.20}$$

i.e. for m = 2 and any odd s, since under these conditions (12.19) contains a term of the form

$$b_j \sin^2 n_0 j\tau ,$$

with a mean value equal to $b_j/2$ and, generally speaking, differing from zero. /165

Thus, additional terms may appear in the averaged system in the resonance case. We shall say that resonance effects are observed in this case.

Functions which are independent of τ may be averaged over the interval of a single period of satellite motion. For instance, let us compute the mean value of the right-hand member of the first equation in system (12.15)

$$A_{1\Omega} = \lim_{T\to\infty} \frac{1}{T} \int_0^T \frac{\sqrt{p}\,\overline{W} \sin u}{R \sin i} d\tau , \tag{12.21}$$

Let us substitute the expression for $\overline{W}$ in (12.21). According to (12.5'), we have

$$\overline{W} = -\frac{R^4}{2\overline{p}^4} \sin 2i \sin u + O(\varkappa). \tag{12.22}$$

Let us take L as the variable of integration in (12.21). Since (12.21) is averaged along the trajectories of undisturbed (Keplerian) motion, it follows from system (12.15) that

$$dL = \tilde{n}_0 \, d\tau .$$

The integrand in (12.21) is a periodic function of L with period 2π, and therefore averaging over a single interval of satellite revolution around the Earth may be substituted for averaging over an infinite interval of motion. Integral (12.21) is transformed to

$$A_{1\Omega} = -\frac{\sqrt{\overline{p}} \cos i}{2\pi \overline{p}^4 \tilde{n}_0} \int_0^{2\pi} R^3 \sin^2 u \, dL. \tag{12.23}$$

Differentiating relationship (12.6), we get

$$\frac{dL}{du} = (1 - e^2)^{3/2} / R^2 . \tag{12.24}$$

With regard to (12.24), integral (12.23) assumes the form

$$A_{1\Omega} = -\frac{\tilde{n} \cos i}{2\pi \overline{p}^2} \int_0^{2\pi} R \sin^2 u \, du. \tag{12.25}$$

Integrating (12.25) and using (12.11),we get

$$A_{1\Omega} = -\frac{\tilde{n}_0 \cos i}{2\overline{p}^2} + O(\varkappa).$$ /166

Integrals of type (12.18) arise during computation of mean values from perturbation due to equatorial flattening. We have already pointed out above that as a consequence of rotation of the Earth and ellipticity of the equator the gravitational field becomes nonconservative and the integral of total energy does not occur. It may be shown by considerations similar to those which were used in calculating integral (12.18) that F remains constant in the nonresonance case. Hence, the energy integral remains constant on the average. In the resonance case, assuming fulfillment of condition (12.20), the total energy of the satellite changes.

Since $\bar{\omega}_3$ is the dimensionless angular velocity of the Earth's rotation, it follows from (12.20) that resonance effects appear if the mean period of revolution of the satellite is equal to or close to 12 hours, 24 hours, 36 hours, etc.

For a satellite with a period of 36 hours or more, disturbance due to the sun and moon is considerably greater than perturbations caused by the ellipticity of the equator (see §1), and therefore, we shall not discuss such orbits.

The mean value of the right-hand members of system (12.15) is calculated analogously to (12.18) and (12.21). We shall give the final form of the averaged system in the first approximation for the nonresonance case:

$$\left.\begin{aligned}
&\frac{d\bar{\Omega}}{d\tau} = -\frac{\varkappa \tilde{n}_0 \cos \bar{i}}{2\bar{p}^2}; \quad \frac{d\bar{i}}{d\tau} = \frac{d\bar{p}}{d\tau} = \frac{d\bar{F}}{d\tau} = 0;\\
&\frac{d\bar{q}}{d\tau} = -\frac{\varkappa \tilde{n}_0 \sqrt{e^2 - \bar{q}^2}}{4\bar{p}^2}(5\cos^2 \bar{i} - 1);\\
&\frac{d\bar{L}}{d\tau} = \tilde{n}_0 + \frac{\varkappa \tilde{n}_0}{4\bar{p}^2}(5\cos^2 i - 1) + 3\varkappa \tilde{n}_0^{1/3}(\bar{F} + U_0).
\end{aligned}\right\} \tag{12.26}$$

Solving the Cauchy problem for (12.26) with initial conditions

$$\tau = 0,\ \bar{\Omega} = \Omega_0,\ \bar{i} = i_0,\ \bar{q} = q_0,\ \bar{p} = 1,\ \bar{F} = 0,\ \bar{L} = L_0,$$

we get

$$\left.\begin{aligned}
&\bar{\Omega} = \Omega_0 - \frac{\tau \varkappa \tilde{n}_0 \cos i_0}{2}; \quad \bar{q} = e_0 \cos \omega, \quad \bar{i} = i_0, \quad \bar{p} = 1;\\
&\bar{F} = 0, \quad L = L_0 + \tau[\tilde{n}_0 + \theta + 3\varkappa n_0^{1/3} U_0].
\end{aligned}\right\} \tag{12.27}$$

Here we use the notation

$$\omega = \theta\tau + \omega_0, \quad \theta = \frac{\varkappa \tilde{n}_0}{4}(5\cos^2 i_0 - 1), \quad \operatorname{tg} \omega_0 = \frac{k_0}{q_0}. \tag{12.28}$$ /167

Having determined the elements of the trajectory from (12.27), we find the argument of the latitude from equation (12.6). The solution found in this way approximates the exact solution of (12.15) with an error of order κ over the time interval $\tau \sim \kappa^{-1}$.

Let us go on to calculation of the resonance case. Let us study the case of principal resonance s = m = 1, where the mean angular velocity of the satellite is equal to or close to the angular velocity of the Earth's rotation. Calculation of the averaged system in this case presents considerable mathematical difficulties. Therefore, we shall limit ourselves to considering only orbits of low eccentricity.

With an error of order $\sim e^3$, the true anomaly ϑ is expressed in terms of the mean anomaly M by the formula [1, 2].

$$\vartheta = M + 2e \sin M + \frac{5}{4} e^2 \sin 2M. \tag{12.29}$$

Going in (12.29) from true anomaly and mean anomaly to the angles u and L according to the formulas

$$u = \vartheta + \operatorname{arctg} \frac{k}{q}; \quad L = M + \operatorname{arctg} \frac{k}{q}, \tag{12.30}$$

we get

$$u = L + 2q \sin L - 2k \cos L + \frac{5}{4}(q^2 - k^2) \sin 2L - \frac{5}{4} qk \cos 2L. \tag{12.31}$$

In place of $\overline{L}$, let us introduce a new variable α -- the phase shift between the mean longitude $\overline{L} + \overline{\Omega}$ of the satellite and the longitude of the semiminor axis of the equatorial ellipse:

$$\alpha = \frac{\pi}{2} + \overline{L} + \overline{\Omega} - \overline{\omega}_3 \tau. \tag{12.32}$$

From system (12.15) we get

$$\begin{aligned} \frac{d\alpha}{d\tau} = \tilde{n}_0 - \overline{\omega}_3 + 3\varkappa \tilde{n}^{1/3} \left(F + U_0 - U - \frac{2\overline{p}}{3R} \frac{\partial U}{\partial \overline{r}} \right) + \\ + \frac{\overline{q}\dot{\overline{k}} - \overline{k}\dot{\overline{q}}}{1 + \sqrt{1 - e^2}} + \dot{\Omega}(1 - \cos i \sqrt{1 - e^2}); \end{aligned} \tag{12.33}$$

$$\frac{d\overline{F}}{d\tau} = \frac{\partial U}{\partial \tau}. \tag{12.34}$$

The angles L and u which appear in the right-hand members of equations (12.33) and (12.34) are expressed in terms of α, Ω and τ by formulas (12.31) and (12.32). /168

In the resonance case, we assume that the difference $\tilde{n}_0 - \overline{\omega}_3$ is a small quantity of order κ. Therefore, $\dot{\alpha} \sim \kappa$ and the angle α, just as the variables $\overline{\Omega}$, $\overline{i}$, $\overline{F}$, $\overline{q}$, $\overline{k}$ in system (12.15) are slowly varying functions of time. The right-hand members of equations (12.33) and (12.34) contain rapidly changing terms which are functions of τ. First-approximation equations are derived by averaging equations (12.33) and (12.34) with respect to τ at fixed values of the slowly changing variables. By averaging equations (12.29) and (12.30), we get the first-approximation system

$$\frac{d\alpha}{d\tau} = \tilde{n}_0 - \overline{\omega}_3 + 3\varkappa\tilde{n}_0^{1/3}(\overline{F}+U_0) + \frac{\varkappa\tilde{n}_0(3\cos^2 i_0 - 1)}{4\overline{p}^2}; \tag{12.35}$$

$$\frac{d\overline{F}}{d\tau} = -\frac{\varkappa b\tilde{n}_0}{8\overline{p}^3}[(2-11e_0^2)(1+\cos i_0)^2 \sin 2\alpha + 9e_0^2 \sin^2 i_0 \sin(2\alpha - 2\omega)]. \tag{12.36}$$

Terms of order $\sim\kappa e^3$ are dropped in (12.32). The averaged equations for $\overline{\Omega}$, $\overline{i}$, $\overline{q}$ and $\overline{p}$ coincide with equations (12.26) for the nonresonance case. Differentiating (12.35) and using (12.26) and (12.36), we get an equation for phase shift

$$\frac{d^2\alpha}{d\tau^2} = -\frac{3\varkappa^2 b}{8}\left(1-\frac{15}{2}e^2\right)[2\sin 2\alpha(1+\cos\overline{i}_0)^2 + 9e^2\sin^2\overline{i}_0 \sin 2(\alpha-\omega)]. \tag{12.37}$$

For orbits of low eccentricity ($e^2 \sim O(\kappa)$), equations (12.37) are simplified:

$$\frac{d^2\alpha}{d\tau^2} = -\frac{3\varkappa^2 b(1+\cos i_0)^2}{4}\sin 2\alpha. \tag{12.38}$$

Solving equations (12.37) and (12.38), we find the function $\alpha(\tau)$, and from formula (12.32) we then find the angle $L(\tau)$. The solutions for $\overline{\Omega}$, $\overline{i}$, $\overline{q}$, $\overline{p}$ in the resonance case coincide with solution (12.27). Finding the function $F(\tau)$ reduces to computing a quadrature. An analytical solution for (12.38) is given in Appendix VIII. Equation (12.37) may also be integrated exactly, just as (12.38), if $i = i_0^* = 63.4°$, $\omega_0 = 0$. In the general case, equation (12.37) should be solved numerically.

Let us formulate the final result of computing satellite motion in the first approximation. In the nonresonance case, the change in elements of the orbit is computed from formulas (12.27), and the value of the argument of latitude is found from equation (12.6). If the period of satellite motion is close to 24 hours, and the orbital eccentricity is of order $O(e^3)$, then the solutions for Ω, i, q and p are taken from (12.27), while the solutions for α, $\overline{F}$, $\overline{L}$ and u are determined from (12.31), (12.32), (12.33) and (12.6). /169

The solution for equation (12.6) may be sought by the method of iterations. If the eccentricity is small ($e \sim O(\kappa^3)$), then the argument of latitude is determined from (12.31).

Let us go on to construction of the second approximation. Subsituting the solution of (12.27) in (12.13), we get a solution for the mean angular velocity

$$\tilde{n} = \tilde{n}_0 + 3\varkappa\tilde{n}_0^{1/3}[F + U_0 - U] + O(\varkappa^2). \qquad (12.39)$$

The functions F, u, q, k and p appearing in (12.39) are multiplied by the small parameter κ. Since the solutions for F, u, q, k and p are found with an error of $\sim\kappa$, solution (12.39) for $\tilde{n}$ has an error of $\sim\kappa^2$. In the nonresonance case F = 0, it follows from (12.39) that $\tilde{n}$ then has no secular or long-period perturbations. This fact was proved previously by Kozai [54].

Let us first examine the nonresonance case. The determination of functions Ω_1, i_1, F_1, q_1, p_1 and L_1 is ambiguous, they are determined to an accuracy which depends on the slowly changing variables $\overline{\Omega}$, $\overline{i}$, $\overline{F}$, $\overline{q}$, $\overline{k}$. In order to eliminate ambiguity, we require that the average values of the functions Ω_1, i_1, F_1, q_1, p_1 and L_1 must be equal to zero with respect to the angle L.

We then get

$$i_1 = \frac{\sin 2i_0}{8}\left[q\cos u - k\sin u + \cos 2u + \frac{q}{3}\cos 3u + \frac{k}{3}\sin 3u\right] - \overline{i}_1; \qquad (12.40)$$

$$\Omega_1 = \frac{\cos i_0}{4}\left[-q\sin u + 3k\cos u + \sin 2u + \frac{q}{3}\sin 3u - \right.$$

$$\left. - \frac{k}{3}\cos 3u\right] + \frac{\overline{\Omega}(u - \overline{L})}{\varkappa\tau\tilde{n}_0} - \overline{\Omega}_1; \qquad (12.41)$$

$$p_1 = 2\,\mathrm{tg}\, i_0 \cdot i_1; \qquad (12.42)$$

/170

$$q_1 = \frac{1}{2}\left\{\left(1+\frac{3k^2+q^2}{4}\right)\cos u+\frac{qk}{6}(\sin 3u-3\sin u)+\frac{q}{2}\cos 2u+\right.$$

$$+\frac{k}{2}\sin 2u+\frac{q^2-k^2}{12}\cos 3u+\frac{5}{4}\sin^2 i_0\left[\cos u\left(-1+\frac{3q^2-13k^2}{10}\right)-\right.$$

$$-\frac{qk}{5}\sin u+\frac{2q}{5}\cos 2u-\frac{6k}{5}\sin 2u+\frac{\cos 3u}{15}\left(7+\frac{11q^2+17k^2}{4}\right)-$$

$$\left.\left.-\frac{qk}{10}\sin 3u+\frac{3q}{10}\cos 4u+\frac{3k}{10}\sin 4u+\frac{q^2-k^2}{20}\cos 5u+\frac{qk}{10}\sin 5u\right]\right\}+$$

$$+k\cos i_0\,\Omega_1^{*}+\frac{k}{4}(1-3\cos^2 i_0)(u-\overline{L})-\overline{q}_1\,; \qquad (12.43)$$

$$L_1=\frac{qk_1-kq_1}{1+\sqrt{1-e_0^2}}-\Omega_1\sqrt{1-e_0^2}\cos i_0+\frac{\sqrt{1-e_0^2}}{2}\left\{q\sin u-k\cos u+\right.$$

$$\left.+\frac{3}{4}\sin^2 i_0\left[\sin 2u+3k\cos u-q\sin u+\frac{q}{3}\sin 3u-\frac{k}{3}\cos 3u\right]\right\}+$$

$$+\frac{\sqrt{1-e_0^2}\,(3\cos^2 i_0-1)}{4}(u-\overline{L})+L_1\,; \qquad (12.44)$$

$$F_1=\Phi(\tau)-\lim_{T\to\infty}\frac{1}{T}\int_0^T\Phi(\tau)\,d\tau\,. \qquad (12.45)$$

Here

$$\Phi(\tau)=\int_0^{\tau}\frac{\partial U(u,\tau)}{\partial\tau}d\tau,\quad q=e_0\cos\omega,\quad k=e_0\sin\omega,\quad \omega=\theta\tau+\omega_0,$$

$$k_1=\frac{1}{2}\left\{\left(1+\frac{3q^2+k^2}{4}\right)\sin u+\frac{q\sin 2u}{2}-\frac{k\cos 2u}{2}-\frac{qk}{6}(\cos 3u+3\cos u)+\right.$$

$$+\frac{q^2-k^2}{12}\sin 3u+\frac{5}{4}\sin^2 i_0\left[-\frac{\sin u}{5}\left(7+\frac{5q^2+9k^2}{2}\right)+\frac{7}{5}qk\cos u+\right.$$

$$+\frac{8}{5}k\cos 2u+\frac{3}{10}qk\cos 3u+\frac{\sin 3u}{15}\left(7+\frac{5q^2+23k^2}{4}\right)-$$

$$\left.\left.-\frac{3k}{10}\cos 4u+\frac{3q}{10}\sin 4u-\frac{qk}{10}\cos 5u+\frac{q^2-k^2}{20}\sin 5u\right]\right\}- \qquad (12.45')$$

$$-q\cos i_0\,\Omega_1+\frac{q(3\cos^2 i_0-1)}{4}(u-\overline{L})-\overline{k}_1\,;$$

$$\bar{\Omega}_1 = \frac{\cos i_0}{8\pi} \int_0^{2\pi} \left[3k \cos u - q \sin u + \sin 2u + \frac{q}{3}\sin 3u - \frac{k}{3}\cos 3u\right] dL =$$ /171

$$= -\frac{e_0^2 \cos i_0 \sin 2\omega}{12} \cdot \frac{1+2\sqrt{1-e_0^2}}{(1+\sqrt{1-e_0^2})^2};$$

$$\bar{i}_1 = \frac{\sin 2i_0}{16\pi} \int_0^{2\pi} \left[q \cos u - k \sin u + \cos 2u + \frac{q}{3}\cos 3u + \frac{k}{3}\sin 3u\right] dL =$$

$$= -\frac{e_0^2 \sin 2i_0 \cos 2\omega}{24} \cdot \frac{1+2\sqrt{1-e_0^2}}{(1+\sqrt{1-e_0^2})^2}.$$

The functions $\bar{q}_1$, $\bar{L}_1$ and $\bar{k}_1$ are calculated analogously to functions $\bar{\Omega}_1$ and $\bar{i}_1$. For $\bar{q}_1$ specifically, we find the following expression:

$$\bar{q}_1 = \frac{q}{6}\left(1 - \frac{3}{2}\sin^2 i_0\right)\left[\frac{5}{3} + \frac{1-e_0^2}{1+\sqrt{1-e_0^2}}\right] + \frac{3q \sin^2 i_0 \cos 2\omega}{16} +$$

$$+ \frac{q}{12}(1-e_0^2)\sin^2 i_0 \cos 2\omega \frac{1+2\sqrt{1-e_0^2}}{(1+2\sqrt{1-e_0^2})^2} - \frac{k \sin 2\omega}{12}\left\{\frac{\cos^2 i_0}{6} \times\right.$$

$$\left.\times \frac{1+2\sqrt{1-e_0^2}}{(1+\sqrt{1-e_0^2})^2} + \sin^2 i_0 \left[\frac{3}{4} + \frac{(1-e_0^2)(1+2\sqrt{1-e_0^2})}{(1+\sqrt{1-e_0^2})^2}\right]\right\}.$$

Functions i_1, Ω_1, p_1, q_1 and L_1 contain short-period disturbances with periods with respect to u equal to 2π, $2\pi/3$, $\pi/2$, $2\pi/5$ and long-period perturbations with periods $2\pi/\theta$, π/θ.

Before going on to construction of the averaged second-approximation system, let us make a number of transformations. The expressions for the square of eccentricity and mean angular velocity are conveniently represented as

$$e^2 = E + \varkappa E_1, \quad \tilde{n} = \bar{n} + \varkappa n_1, \tag{12.46}$$

where

$$\bar{n} = \tilde{n}_0 + 3\varkappa \tilde{n}_0^{1/3}(F + U_0 - \bar{U}); \; E = 1 - \bar{p}\bar{n}^{2/3}; \; \bar{U} = \frac{(1-e_0^2)^{3/2}}{6}\left(1 - \frac{3}{2}\sin^2 i_0\right).$$

From (12.42) and (12.43), we find

$$E_1 = -p_1 \bar{n}^{2/3} - \frac{2}{3} n_1 \bar{n}^{-1/3}. \tag{12.47}$$

We then have /172

$$\bar{E}_1 = \frac{1}{2\pi} \int_0^{2\pi} E_1 d\bar{L} = -2(1-E)\operatorname{tg} i_0 \bar{i}_1.$$

In order to compute the averaged second-approximation system, it is necessary to find the functions

$$\left(\frac{R}{\bar{p}}\right)_1 = \frac{\partial\left(\frac{R}{\bar{p}}\right)}{\partial \bar{p}} p_1 + \frac{q_1}{\bar{p}} \frac{\partial R}{\partial q} + \frac{k_1}{\bar{p}} \frac{\partial R}{\partial k} + \frac{L_1}{\bar{p}} \frac{\partial R}{\partial L}; \tag{12.48}$$

$$u_1 = \frac{\partial u}{\partial L} L_1 + \frac{\partial u}{\partial L} \cdot \frac{\partial L}{\partial q} q_1 + \frac{\partial u}{\partial L} \cdot \frac{\partial L}{\partial k} k_1. \tag{12.49}$$

According to (12.27), $\bar{p} = 1 + O(\kappa)$, and therefore

$$\left(\frac{R}{\bar{p}}\right)_1 = -Rp_1 + q_1 \frac{\partial R}{\partial q} + k_1 \frac{\partial R}{\partial k} + L_1 \frac{\partial R}{\partial L} + O(\varkappa). \tag{12.50}$$

Substituting formulas (12.7), (12.42), (12.43) and (12.44) in (12.49) and (12.50), we get

$$\begin{aligned} \left(\frac{R}{\bar{p}}\right)_1 = \frac{R^2}{6}\Bigg\{&\left(1-\frac{3}{2}\sin^2 i\right)\left[1+\frac{e\cos\vartheta}{1+\sqrt{1-e^2}}+\frac{2}{R}\sqrt{1-e^2}\right]-\frac{1}{2}\sin^2 i\cos 2u + \\ &+ \frac{3e\sin^2 i}{2(1-e^2)}\sin 2\omega\sin\vartheta\left[1-\frac{2+e^2}{3}\,\frac{1+2\sqrt{1-e^2}}{(1+\sqrt{1-e^2})^2}\right] - \\ &- \frac{e\sin^2 i\cos 2\omega}{2}\,\frac{1+2\sqrt{1-e^2}}{(1+\sqrt{1-e^2})^2}\Bigg\}; \end{aligned} \tag{12.51}$$

$$\begin{aligned} u_1 = \frac{4-5\sin^2 i}{4}(u-L) + \frac{1}{12}\Bigg\{&\left(1-\frac{3}{2}\sin^2 i\right)\left[\frac{4}{e}\sin\vartheta\left(1-\sqrt{1-e^2}-\frac{e^2}{2}\right)+\right. \\ &\left.+\sin\vartheta(1-\sqrt{1-e^2})\right] - 3e\sin(\vartheta+2\omega)\left(1-\frac{5}{3}\sin^2 i\right) - \\ &-3\sin 2u\left(1-\frac{7}{6}\sin^2 i\right) - e\cos^2 i\sin(3\vartheta+2\omega) + 12e\sin\vartheta\left(1-\frac{5}{4}\sin^2 i\right) + \\ &+\frac{3}{2}\sin^2 i\sin 2\omega - \frac{3R^2\sin^2 i\sin 2\omega}{2(1-e^2)} + \frac{1+2\sqrt{1-e^2}}{(1+\sqrt{1-e^2})^2} \times \end{aligned}$$

$$\times\left[\frac{2+e^2}{2(1-e^2)}R^2\sin^2 i\sin 2\omega - e^2\cos^2 i\sin 2\omega - \right. \tag{12.51}$$

$$\left. - e(1+R)\sin\vartheta\sin^2 i\cos 2\omega + (e^2-1)\sin^2 i\sin 2\omega\right], \tag{12.52}$$

Here $i = i_0$, $e = e_0$, $\omega = \theta\tau + \omega_0$. /173

Let us illustrate some singularities in computing the averaged system by example of the integral

$$I_3 = \frac{\theta\cos i_0}{2\pi(1-e^2)^{3/2}}\int_0^{2\pi} R^3\sin 2u\cdot(\bar{L}-u)\,d\bar{L}. \tag{12.53}$$

We use the notation I_4 to designate the integral

$$I_4 = -\frac{\cos i_0}{(1-e^2)^{3/2}}\int_0^{2\pi} R^3\sin 2u\,d\bar{L} = \frac{\cos i_0}{2}\left[\cos 2u + e\cos(u+\omega) + \frac{e}{3}\cos(3u+\omega)\right]. \tag{12.54}$$

We represent integral (12.53) in the form

$$I_3 = \frac{\theta}{2\pi}\int_0^{2\pi}\frac{\partial I_4}{\partial u}(u-\bar{L})\,d\bar{L}. \tag{12.55}$$

Integrating (12.55) by parts, we get

$$I_3 = -\frac{\theta}{2\pi}\int_0^{2\pi} I_4\,du + \frac{\theta}{2\pi}\int_0^{2\pi} I_4\,d\bar{L}. \tag{12.56}$$

It follows from the form of the integrand in I_4 (12.54) that the first integral in (12.56) is equal to zero. The second integral may be written in the form

$$\frac{\theta}{2\pi}\int_0^{2\pi} I_4\,d\bar{L} = \frac{\partial\bar{\Omega}_1}{\partial\tau} = I_3\,.$$

Omitting intermediate computations, we give the final form of the averaged system in the second approximation:

$$\frac{di}{d\tau} = -\varkappa^2 EH \sin i_0 \cos i_0 \sin 2\omega -$$

$$- \frac{3}{8}\varkappa^2 n f\sqrt{E}\cos i_0 \cos\omega(5\cos^2 i_0 - 1) + \varkappa\frac{di_1}{d\tau} = A_i\,; \tag{12.57}$$

$$\frac{d\bar\Omega}{d\tau} = -\frac{\varkappa\tilde n_0 \cos\bar i}{2\bar p^2} - \varkappa^2 G - \varkappa^2 DE\cos 2\omega - \frac{5\varkappa^2}{2}\tilde n_0 \bar i_1 \sin i_0 +$$

$$+ \frac{3\varkappa^2}{8}\tilde n_0 f\sqrt{E}\,\mathrm{ctg}\, i_0 \sin\omega(15\sin^2 i_0 - 4) + \varkappa\frac{d\bar\Omega_1}{d\tau}\,; \tag{12.58}$$

$$\frac{d\bar p}{d\tau} = 2\,\mathrm{tg}\, i_0 \cdot A_i\,; \tag{12.59}$$

$$\frac{d\bar q}{d\tau} = -\frac{\varkappa\tilde n_0 \bar k}{4\bar p^2}(5\cos^2\bar i - 1) - \varkappa^2\bar k[A + G\cos i_0 - ED\cos i_0] + \varkappa\frac{d\bar q_1}{d\tau} -$$

$$- \bar k\varkappa^2\bar q^2[7H\sin^2 i_0 + 2D\cos i_0] + \frac{\varkappa^2\tilde n_0}{8\bar k}(5\cos^2 i_0 - 1)(2\bar q\bar q_1 - \bar E_1) -$$

$$- \frac{3\varkappa^2 k^2 f\tilde n_0}{8E\sin i_0}[(1+4E)(5\sin^2 i_0 - 4)\sin^2 i_0 - E\cos^2 i_0(15\sin^2 i_0 - 4)] +$$

$$+ \frac{3\varkappa^2}{8E}\tilde n_0 \bar q^2 f(1-E)(5\cos^2 i_0 - 1) - \frac{\varkappa^2\tilde n_0}{2}\bar k\,\mathrm{tg}\, i \cdot \bar i_1(15\cos^2 i_0 - 2); \tag{12.60}$$ /174

$$\frac{d\bar L}{d\tau} = \tilde n_0 + \frac{\bar q\dot E - 2\dot{\bar q}E}{2\bar k(1+\sqrt{1-E})} - \dot{\bar\Omega}\sqrt{1-E}\cos\bar i + \varkappa N + \varkappa^2 M\cos 2\omega + \frac{\varkappa\tilde n\sqrt{1-E}}{2\bar p^2}\times$$

$$\times\left(1 - \frac{3}{2}\sin^2\bar i\right) + \frac{15\varkappa^2}{8}\tilde n_0 f\sqrt{E(1-E)}\sin i_0 \sin\omega(1 - 5\cos^2 i_0) +$$

$$+ \tilde n_0\varkappa^2(5\cos^2 i_0 - 1)\,\mathrm{tg}\, i_0\sqrt{1-E}\;\bar i_1\,; \tag{12.61}$$

Here

$$\frac{dE}{d\tau} = -2(1-E)\,\mathrm{tg}\,\bar i \cdot A_i\,; \tag{12.62}$$

$$\bar k = \sqrt{E - \bar q^2}\,;\quad \mathrm{tg}\,\omega = \frac{\bar k}{\bar q}\,;\quad H = \frac{\tilde n_0}{96}[15\sin^2 i_0 - 14 + 18C(6 - 7\sin^2 i_0)];$$

$$G = \frac{\tilde n_0 \cos i_0}{4}\left[\frac{3}{2} + \frac{E}{6} + \sqrt{1-E} + \sin^2 i_0\left(-\frac{5}{3} + \frac{5}{24}E - \frac{3}{2}\sqrt{1-E}\right) +\right.$$

$$+\frac{6}{1-e_0^2}(U_0-\bar{U})+\frac{3c}{2}\left(1+\frac{3}{2}E\right)(4-7\sin^2 i_0)\Big];$$

$$D=\frac{\tilde{n}_0\cos i_0}{16}\left[\frac{7}{3}-5\sin^2 i_0+6c(7\sin^2 i_0-3)\right];$$

$$A=\tilde{n}_0\Big\{\left(1-\frac{3}{2}\sin^2 i_0\right)^2\left(\frac{5}{24}+\frac{\sqrt{1-E}}{4}\right)+\frac{3(5\cos^2 i_0-1)}{4(1-e_0^2)}(U_0-\bar{U})+$$

$$+\frac{5}{48}\left[4+E-2\sin^2 i_0+\frac{5}{2}E\sin^2 i_0-\frac{15}{2}\sin^4 i_0-\right.$$

$$\left.-\frac{35}{8}E\sin^4 i_0\right]-\frac{3c}{32}\Big[(6-7\sin^2 i_0)(2+5E)\sin^2 i_0-$$

$$-4(4+3E)\left(1-5\sin^2 i_0+\frac{35}{8}\sin^4 i_0\right)\Big]\Big\};$$

$$N=3\tilde{n}_0^{1/3}(F+U_0)\left[1+\frac{\varkappa U_0}{2(1-e_0^2)}\right]+\frac{\varkappa\tilde{n}_0(1-E)}{4}\left(1-\frac{3}{2}\sin^2 i_0\right)^2+$$ /175

$$+\frac{7\tilde{n}_0\varkappa}{8\sqrt{1-E}}\Big\{\frac{2}{3}-\frac{E}{2}-\frac{E^2}{6}+\sin^2 i_0\Big[-\frac{3}{2}+\frac{4}{3}E+\frac{E^2}{6}+$$

$$+5\sin^2 i_0\left(\frac{1}{6}-\frac{3}{16}E+\frac{E^2}{48}\right)\Big]+\frac{3c(2+3E)(1-E)}{5}\times$$

$$\times\left(1-5\sin^2 i_0+\frac{35}{8}\sin^4 i_0\right)\Big\};$$

$$M=\frac{7\tilde{n}_0 E\sqrt{1-E}\sin^2 i_0}{192}[15\sin^2 i_0-14+18c(6-7\sin^2 i_0)].$$

With regard to (12.16) and (12.27), we write the initial conditions for (12.57)-(12.61):

$$\left.\begin{aligned}&\bar{\Omega}=\Omega_0-\varkappa\Omega_{10};\quad \bar{i}=i_0-\varkappa i_{10};\quad \bar{q}=q_0-\varkappa q_{10};\\&\bar{p}=1-\varkappa p_{10};\quad \bar{L}=L_0-\varkappa L_{10}.\end{aligned}\right\}\qquad(12.63)$$

We substitute solutions (12.27) from the first-approximation equations for the functions $\bar{q}$, $\bar{k}$, $\bar{i}$, $\bar{p}$ and $\bar{F}$ in the right-hand members of equations (12.57), (12.59) and (12.62). Since the right-hand members of equations (12.57), (12.59) and (12.62) are proportional to κ^2 and approximate solution (12.27) is found with an error of $\sim\kappa$, the equations obtained after substituting solutions (12.27) in place of $\bar{q}$, $\bar{k}$, $\bar{i}$, $\bar{p}$ and $\bar{F}$ will be computed with an error of $\sim\kappa^3$, but initial conditions (12.15) were constructed with this same error. Therefore, substitution (12.27) is valid. Calculation of $\bar{i}$, $\bar{p}$ and E now reduces to quadratures. We have finally

$$\bar{i} = i_0 + \frac{\varkappa^2 e_0^2 H}{4\theta} \sin 2i_0 [\cos 2\omega - \cos 2\omega_0] - \frac{3}{2}\varkappa e_0 f \cos i_0 \times$$
$$\times [\sin\omega - \sin\omega_0] + \varkappa \bar{i}_1 - \varkappa i_{10} - \varkappa \bar{i}_{10}; \tag{12.64}$$
$$E = e_0^2 - 2(1-e_0^2)\,\mathrm{tg}\, i_0 (\bar{i} - i_0) + 2\varkappa (U - U_0); \tag{12.65}$$
$$\bar{p} = 1 + 2\varkappa\, \mathrm{tg}\, i_0 (\bar{i} - i_0). \tag{12.66}$$

We write the solution for equation (12.60) in the form

$$\bar{q} = \sqrt{E}\cos(\theta\tau + \omega_1) + \varkappa(\varphi + q_1), \tag{12.67}$$

where /176

$$\mathrm{tg}\,\omega_1 = \frac{k_0 - \varkappa k_{10}}{q_0 - \varkappa q_{10}}.$$

Differentiating (12.67) in the light of system (12.57)-(12.61), we get an equation for the new variable φ. We substitute solutions (12.64)-(12.66) in the right-hand members of the system of equations for $\bar{\Omega}$, $\bar{L}$ and ϕ, and expand them in a series with respect to powers of κ. We drop terms of order κ^3 in the equations for $\bar{\Omega}$ and $\bar{L}$, and retain only terms of the first negative order of magnitude in the equations for φ. We then get

$$\left.\begin{aligned}
\frac{d\Omega}{d\tau} &= -\frac{\varkappa\tilde{n}_0}{2}\cos i_0 + \frac{5}{2}\varkappa\tilde{n}_0 \sin i_0 (\bar{i} - i_0) + \varkappa\Phi_1;\\
\frac{d\varphi}{d\tau} &= \Phi_2 + \frac{\varkappa\tilde{n}_0\,\mathrm{tg}\, i_0}{2}(2 - 15\cos^2 i_0)(\bar{i} - i_0) +\\
&\quad + \theta\varphi\,\mathrm{ctg}(\theta\tau + \omega_1) - \frac{E}{2\varkappa\sqrt{E}}\cos(\theta\tau + \omega_1) - \dot{\bar{q}}_1;\\
\frac{d\bar{L}}{d\tau} &= \tilde{n}_0 + \varkappa\dot{\bar{L}}_1 - \dot{\bar{\Omega}}\sqrt{1-e_0^2}\cos i_0 + \varkappa N + \varkappa^2 M\cos 2\omega + \frac{\theta E + \varkappa\dot{\varphi}\sqrt{E}}{1+\sqrt{1-E}} +\\
&\quad + \frac{\varkappa\bar{n}\sqrt{1-e_0^2}}{2}\left(1 - \frac{3}{2}\sin^2 i_0\right) - \frac{15}{2}\varkappa\theta f e_0\sqrt{1-e_0^2}\sin i_0 \sin\omega -\\
&\quad - 2\varkappa^2 H e_0^2\sqrt{1-e_0^2}(\cos 2\omega - \cos 2\omega_0) + 6\theta\varkappa f e_0\sqrt{1-e_0^2}\times\\
&\quad \times \sin i_0(\sin\omega - \sin\omega_0) + 4\theta\varkappa\,\mathrm{tg}\, i_0\sqrt{1-e_0^2}(i_{10} + \bar{i}_{10}).
\end{aligned}\right\} \tag{12.68}$$

Determination of the functions $\bar{\Omega}$ and $\bar{L}$ has now been reduced to quadratures. The function φ satisfies a linear differential equation. Solving (12.68) with initial conditions (12.63), we get

$$\begin{aligned} \bar{\Omega} = {} & \Omega_0 - \varkappa\Omega_{10} + \varkappa\bar{\Omega}_1 - \varkappa\bar{\Omega}_{10} - \frac{\varkappa\tau\tilde{n}_0}{2}\cos i_0 - \frac{\varkappa^2 e_0^2 D}{2\theta}(\sin 2\omega - \\ & - \sin 2\omega_0) - \varkappa^2\tau\left[G + \frac{5}{2}\tilde{n}_0 \sin i_0 (i_{10} + \bar{i}_{10})\right] + \frac{3}{2}\varkappa e_0 f \operatorname{ctg} i_0 \times \\ & \times \left[\cos\omega - \cos\omega_0 + \frac{5}{2}\varkappa\tau\tilde{n}_0 \sin^2 i_0 \sin\omega\right] + \frac{5e_0^2\varkappa\tilde{n}_0}{8\theta^2} H \sin^2 i_0 \cos i_0 \times \\ & \times (\sin 2\omega - \sin 2\omega_0 - 2\theta\tau\cos 2\omega_0); \end{aligned} \tag{12.69}$$

$$\begin{aligned} \frac{\varphi}{\sin(\theta\tau+\omega_0)} = {} & \tau e_0\varkappa^2\Big\{A + G\cos i_0 - e_0^2 D\cos i_0 + \frac{\tilde{n}_0}{2}\operatorname{tg} i_0 (15\cos^2 i_0 - 2) \times \\ & \times\left[i_{10} + \bar{i}_{10} + \frac{\varkappa q_0^2}{2\theta} H \sin 2i_0\right]\Big\} + \frac{\varkappa^2 e_0}{4\theta}[\sin 2\omega - \sin 2\omega_0 + 2\theta\tau] \times \\ & \times\Big[2e_0^2 D\cos i_0 + H(2+5e_0^2)\sin^2 i_0 - \frac{\tilde{n}_0 e_0^2}{2\theta} H(15\cos^2 i_0 - \\ & - 2)\sin^2 i_0\Big] + \frac{3\varkappa f}{2\sin i_0}\Big\{(\sin^2 i_0 - e_0^2\cos^2 i_0)(\cos\omega - \\ & - \cos\omega_0) - \frac{\varkappa}{2}\tilde{n}_0 e_0^2 \sin^2 i_0 \sin^2\omega_0 \cdot (15\cos^2 i_0 - 2)\Big\}; \end{aligned} \tag{12.70}$$

/177

$$\begin{aligned} \bar{L} = {} & \tilde{n}_0\tau + \frac{\varkappa e_0\varphi + \theta\tau e_0^2}{1+\sqrt{1-e_0^2}} - (\bar{\Omega} - \Omega_0)\sqrt{1-e_0^2}\cos i_0 + \varkappa\bar{L}_1 - \varkappa L_{10} - \varkappa\bar{L}_{10} + \\ & + \frac{\varkappa\tau}{2}(1-e_0^2)^2\left(1 - \frac{3}{2}\sin^2 i_0\right)\left(1 + \frac{3\varkappa(U_0 - \bar{U})}{1-e_0^2}\right) + \\ & + \frac{\varkappa e_0[k_1\cos(\theta\tau+\omega_0) - \bar{q}_1\sin(\theta\tau+\omega_0)]}{1+\sqrt{1-e_0^2}} + \frac{\varkappa^2 M}{2\theta}(\sin 2\omega - \sin 2\omega_0) - \\ & - \frac{\varkappa^2 e_0^2 H}{\theta}\sqrt{1-e_0^2}\sin^2 i_0 [\sin 2\omega - \sin 2\omega_0 - 2\tau\theta\cos 2\omega_0] + \\ & + \frac{3}{2}\varkappa e_0 f\sqrt{1-e_0^2}\sin i_0 [\cos\omega - \cos\omega_0 - 4\theta\tau\sin\omega_0]. \end{aligned} \tag{12.71}$$

The components of the Laplace vector are computed from the formulas

$$\left.\begin{aligned} q &= \sqrt{E}\cos(\theta\tau+\omega_1) + \varkappa(\varphi + q_1 + \bar{q}_1); \\ k &= \sqrt{E}\sin(\theta\tau+\omega_1) + \varkappa[-\varphi\operatorname{ctg}(\theta\tau+\omega_0) + k_1 + \bar{k}_1]. \end{aligned}\right\} \tag{12.72}$$

Let us give the final rule for calculating satellite motion in the second approximation. After first-approximation solutions have been found for the functions $\bar{\Omega}$, $\bar{L}$, $\bar{p}$, $\bar{F}$, $\bar{i}$, $\bar{q}$ and u, we substitute their values in (12.40)-(12.44) and determine $\bar{i}$, $\bar{p}$, $\bar{\Omega}$, E, q, k and $\bar{L}$ by formulas (12.64) (12.66) and (12.69)-(12.71). Returning then to the initial variables, we find the functions i, $\tilde{p}$, Ω and L. The argument of latitude in the second approximation is found from equation (12.6). The final result is independent of functions $\bar{\Omega}_1$, $\bar{i}_1$, $\bar{p}_1$, $\bar{q}_1$, $\bar{k}_1$ and $\bar{L}_1$.

The constructed solution approximates the exact solution of system (12.5) with an error of $\sim\kappa^2$ on the interval $\tau \sim \kappa^{-1}$. This estimate of accuracy holds if $\tilde{n} \sim 1$. For orbits of greater eccentricity, $\tilde{n}$ is small, and the approximate solution in this case has an error of $\sim\tilde{n}^{-2}\kappa^2$ on the interval $\tau \sim \kappa^{-1}$. /178

The solution found for i, $\tilde{p}$ coincides with Kozai's solution [54]. The solutions for Ω, q, k and L differ from the solutions in [54].

In the case of equatorial orbits, the angles $\Omega + \omega$, $L + \omega$ should be substituted for the angles Ω, L. The effect of the third zonal harmonic on the motion of an equatorial satellite requires special consideration, and we shall not take up this problem.

Calculation of satellite motion by approximate formulas involves two difficulties: first the solution of equation (12.6), and secondly calculation of the quantity F_{10}. However, since solution of (12.6) is equivalent to solution of Kepler's equation, methods may be applied here which are ordinarily used in celestial mechanics (method of iterations, gradient method, expansion in series). In the case of orbits with low eccentricity, the argument of latitude is computed by the explicit formula

$$u = L + 2e_0(1-e_0^2)\sin(L-\omega) + \frac{5}{4}e_0^2\left(1-\frac{11}{30}e_0^2\right)\sin 2(L-\omega) +$$
$$+ \frac{13}{12}e_0^3\sin 3(L-\omega) + \frac{103}{96}e_0^4\sin 4(L-\omega) + \dots .$$

For computing F_{10}, we use integration by parts to transform formula (12.45) to the form

$$F_{10} = \lim_{T\to\infty}\left[\frac{1}{T}\int_0^T \tau\frac{\partial U}{\partial\tau}d\tau - \int_0^T\frac{\partial U}{\partial\tau}d\tau\right]. \qquad (12.73)$$

Taking the argument of latitude as the variable of integration, we get

$$F_{10} = \lim_{u_1 \to \infty} \left[\frac{1}{u_1} \int_{u_0}^{u_1} \frac{\tau}{R^2} \frac{\partial U}{\partial \tau} du - \int_{u_0}^{u_1} \frac{1}{R^2} \frac{\partial U}{\partial \tau} du \right] + O(\varkappa). \tag{12.74}$$

Thus, the determination of F_{10} reduces to quadratures which may be done numerically, or in the case of orbits of sufficiently low eccentricity, may be found analytically by series expansions. The extent of the integration interval u_1 in (12.74) should be chosen so that the error in computing F_{10} does not exceed κ. /179

In formulas (12.45), (12.73) and (12.74), the symbol $\partial/\partial\tau$ denotes differentiation with respect to time given explicitly at a fixed value of u. Integration in (12.45), (12.73) and (12.74) is based on the assumption that u is a function of τ. The relationship $\tau(u)$ is found from (12.6), assuming $q = q_0$, $k = k_0$, $L = \tilde{n}_0 + L_0$.

In the expression for the potential of disturbing forces (12.2), two tesseral harmonics were previously retained, but since only the quantity F_{10} depends on tesseral harmonics in the second approximation, the effect of these harmonics may be easily accounted for if the remaining tesseral harmonics are included in integrals (12.74) in the expression for U.

When a computer is used in calculating satellite motion, it is convenient to use the method of iterations for solving equation (12.6), and F_{10} should be calculated from the known quadrature formulas. The use of approximate formulas results in a sharp reduction of machine time in calculating satellite orbits.

Computation may be simplified somewhat by using the formulas

$$e^2 = e_0^2 - 2(1-e_0^2)\operatorname{tg} i_0(\bar{i}-i_0) + 2\varkappa(U-U_0) + \varkappa E_1; \tag{12.75}$$

$$\bar{k} = \sqrt{E}\sin(\theta\tau+\omega_1) - \varkappa\varphi \operatorname{ctg}(\theta\tau+\omega_0). \tag{12.76}$$

Formula (12.75) follows from (12.46) and (12.65). The second-approximation solution is now constructed as follows: After calculating the first approximation from formulas (12.40)-(12.43), (12.64)-(12.67), (12.69)-(12.73) and (12.16), we determine Ω, i, p, q and e; we then find k from (12.76). We compute k_1 from the formula

$$k_1 = \sqrt{e^2 - q^2} - \bar{k}. \qquad (12.77)$$

We then find L from (12.44), (12.71) and (12.16). While this procedure eliminates the use of (12.45') in calculating k_1, it is not applicable if k is small, since in this case the difference between close quantities $q^2 - k^2$ in (12.77) has a large relative error. Therefore, in order to avoid loss of accuracy in determining q and k, they should be calculated by formulas (12.72).

Besides using the resultant solution for the purely computational purpose of predicting satellite motion, it may be used for qualitative analysis of the effect which the Earth's oblateness has on satellite motion. It is extremely convenient that variables ordinarily used in celestial mechanics are used in the resultant solution. For this reason, disturbed motion may be qualitatively studied directly on the basis of the approximate solution without preliminary transformations. /180

It is obvious from the solution that the disturbance of Keplerian motion is extremely complex in the general case. Out of all the elements, we should isolate inclination, the square of eccentricity, mean angular velocity and the focal parameter. The solution for these may be written in the form

$$p = 1 + 2\,\mathrm{tg}\, i_0 (i - i_0); \quad \tilde{n} = \tilde{n}_0 + 3\varkappa \tilde{n}_0^{1/3} (U_0 - U);$$

$$e^2 = e_0^2 - 2(1 - e_0^2)\,\mathrm{tg}\, i_0 (i - i_0) - \frac{2}{3} \tilde{n}_0^{-1/3} (\tilde{n} - \tilde{n}_0).$$

Hence, it follows that long-period perturbations of the functions p, i, e^2, $-\frac{2}{3}\tilde{n}^{-1/3}(\tilde{n} - \tilde{n}_0)$ differ only by constant multiples. The short-period perturbations of these functions are similarly related. The elements p, i, e and $\tilde{n}$ do not undergo any secular variations. Disturbances of Ω, L and ω are much more complex.

It follows from the resultant solution that the tesseral harmonics of the Earth's gravitational potential in the second approximation have no effect on orbital elements Ω, i, p, q and k; however, a secular disturbance of the angle L shows up, which changes the position of the satellite in the orbit. Therefore, if it is necessary to calculate only the evolution of the satellite's orbit, while its position in the orbit is not consequential, then tesseral harmonics may be disregarded in the nonresonance case.

The approximate solution is simplified in the case of nearly circular orbits, where $e_0 \sim O(\varkappa)$. In this case, by dropping terms $\sim\varkappa$ from (12.40)-(12.45), we find an expression for short-period disturbances

$$\left.\begin{aligned}
&\Omega_1 = \frac{1}{4}\cos i_0 \sin 2u, \quad i_1 = \frac{1}{2}\sin 2i_0 \cos 2u;\\
&q_1 = \frac{1}{2}\left[\cos u - \frac{5}{4}\sin^2 i_0 \cos u + \frac{7}{12}\cos 3u \cdot \sin^2 i_0\right];\\
&k_1 = \frac{1}{2}\left[\sin u - \frac{7}{4}\sin^2 i_0 \sin u + \frac{7}{12}\sin 3u \sin^2 i_0\right];\\
&p_1 = \frac{1}{2}\sin^2 i_0 \cos 2u, \quad L_1 = \frac{1}{4}\sin 2u\left[\frac{5}{2}\sin^2 i_0 - 1\right].
\end{aligned}\right\} \tag{12.78}$$

From (12.64)-(12.66) and (12.69)-(12.71), we get the expressions for secular and long-period disturbances of orbital elements: /181

$$\left.\begin{aligned}
&\bar{\Omega} = \Omega_0 - \varkappa\Omega_{10} - \frac{\varkappa\tau}{2}\cos i_0\left\{1 + \frac{\varkappa}{4}\left[5 - \frac{19}{3}\sin^2 i_0 + \right.\right.\\
&\qquad \left.\left. + 3c(4 - 7\sin^2 i_0) + 12(U_0 - \bar{U}) + 5\sin^2 i_0 \cos 2u_0\right]\right\};\\
&\bar{i} = i_0 - \varkappa i_{10}, \quad \bar{p} = 1 - \varkappa p_{10};\\
&\bar{k} = \frac{3}{2}\varkappa f \sin i_0 + \frac{\delta}{2}\sin\left\{\theta\tau + \arcsin\frac{1}{\delta}[2k_0 - 2\varkappa k_{10} - \right.\\
&\qquad\qquad \left. - 3\varkappa f \sin i_0]\right\};\\
&\bar{q}^2 + \bar{k}^2 = E_0 + 3\varkappa f \sin i_0 (k - k_0 + \varkappa k_{10});\\
&\bar{L} = \tau\left\{\tilde{n}_0 + \varkappa N + \varkappa\theta\sin^2 i_0 \cos 2u_0 + \frac{\varkappa\tilde{n}_0}{4}(3\cos^2 i_0 - 1) \times \right.\\
&\qquad \left. \times (1 + 3\varkappa U_0)\right\} - \bar{\Omega}\cos i_0 - \varkappa L_{10}.
\end{aligned}\right\} \tag{12.79}$$

Here

$$N = 3(U_0 - \varkappa F_{10})\left(1 + \frac{\varkappa}{2}U_0\right) + \frac{\varkappa}{4}\left[\frac{10}{3} - \frac{33}{4}\sin^2 i_0 + \frac{31}{6}\sin^4 i_0 + \right.$$
$$\left. + \frac{21}{5}c\left(1 - 5\sin^2 i_0 + \frac{35}{8}\sin^4 i_0\right)\right];$$

$$F_{10} = \frac{b}{8}\left[2\sin^2 i_0 \cos 2\Omega_0 + \frac{\tilde{\omega}_3(1 - \cos i_0)^2}{1 + \tilde{\omega}_3}\cos 2(u_0 - \Omega_0) + \right.$$
$$\left. + \frac{\tilde{\omega}_3(1 + \cos i_0)^2}{\tilde{\omega}_3 - 1}\cos 2(u_0 + \Omega_0)\right];$$

$$\delta^2 = 4[E_0 + 3\varkappa f \sin i_0(\varkappa k_{10} - k_{10})];$$

$$E_0 = (q_0 - \varkappa q_{10})^2 + (b_0 - \varkappa b_{10})^2.$$

The approximate solution found from formulas (12.16), (12.78) and (12.79) is a solution of the second approximation: it approximates the exact solution of (12.5) with error $\sim \kappa^2$ over the interval $\tau \sim \kappa$, if $e_0 \sim 0(\kappa)$.

Using solution (12.78), (12.79) and assuming f = 0, we find the formula for the dimensionless focal radius of the orbit

/182

$$\bar{r} = 1 - e_0 \cos(\theta\tau + \omega_0 - u) + \frac{\varkappa}{2}\left\{\left(1 - \frac{5}{4}\sin^2 i\right)\cos(\theta\tau + u_0 - u) - 1 + \right.$$
$$+ \frac{7}{12}\sin^2 i_0 \cos(\theta\tau + 3u_0 - u) + \frac{\sin^2 i_0}{6}\cos 2u - \sin^2 i_0 \cos 2u_0 + \tag{12.80}$$
$$\left. + \frac{3}{2}\sin^2 i_0 + \frac{1}{2}\sin^2 i_0 \sin^2 u_0 \sin(\theta\tau - u)\right\}.$$

The solution for the problem is simplified if the orbital inclination is close to $i_* = 63.4°$. In this case, assuming that the difference $5 \cos^2 i_0 - 1$ is a quantity of the first negative order of magnitude, from (12.64)-(12.66) and (12.69)-(12.71) we get formulas for long-period and secular disturbances of the orbital elements:

$$\bar{i} = i_0 - \varkappa i_{10} - \frac{2}{5}\varkappa^2 e_0^2 \tau H \sin 2\omega_0; \quad \bar{p} = 1 + 4(\bar{i} - i_0);$$
$$\Omega = \Omega_0 - \varkappa\Omega_{10} - \varkappa\tau\left[\frac{\tilde{n}_0}{2}\cos i_0 + \varkappa G + \varkappa D e_0^2 \cos 2\omega_0 - \frac{3}{2}\varkappa\tilde{n}_0 e_0 f \sin\omega_0 + \right.$$
$$\left. + \frac{5}{2}\varkappa\tilde{n}_0 \sin i_0 (i_{10} + \bar{i}_1)\right] - \frac{\varkappa^3}{2} e_0^2 \tau^2 \tilde{n}_0 H \sin i_0 \sin 2\omega_0;$$
$$\varphi = -\tau\varkappa e_0\left\{A + G\cos i_0 - e_0^2 D \cos i_0 + \frac{3}{2}k_0 \tilde{n}_0 f \cos i_0 - \tilde{n}_0 i_{10} - \tilde{n}_0 \bar{i}_1 + \right.$$
$$\left. + \cos^2\omega_0\left[2De_0^2 \cos i_0 + \frac{4}{5}H(2 + 5e_0^2)\right] - \frac{\tilde{n}_0}{5}\varkappa e_0^2 \tau H \sin 2\omega_0\right\}\sin\omega_0;$$
$$\bar{L} = L_0 - \varkappa L_{10} + \tilde{n}_0\tau + \frac{\varkappa\varphi e_0 + \tau\theta e_0^2}{1 + \sqrt{1 - e_0^2}} + (\Omega_0 - \bar{\Omega})\sqrt{1 - e_0^2}\cos^2 i_0 + \varkappa\tau\left[N + \right.$$
$$\left. + \varkappa M \cos 2\omega_0 - \frac{\tilde{n}_0}{10}\sqrt{1 - e_0^2}\,(1 + 3\varkappa\tilde{n}_0^{-3/4}(U_0 - \bar{U}))\right] + \frac{\bar{q}k_1 - \bar{k}q_1}{1 + \sqrt{1 - e_0^2}}.$$

We substitute $q = q_0$, $k = k_0$ in the expressions for i_1, Ω_1, p_1, q_1, k_1 and L_1.

The formulas derived above may be used for approximate calculation of the evolution of satellite motion. An important advantage of the resultant solutions is the fact that the solution is found in explicit form for osculating variables ordinarily used in celestial mechanics. Therefore, it may be used for qualitative investigation of disturbed satellite motion without additional transformations.

§13. Hyperelliptic Theory of Satellite Motion /183

A form of potential function may be found which approximates the Earth's gravitational field fairly well, and a system of coordinates may be found for which the equations of disturbed motion are integrated in quadratures. The problem of motion of an artificial Earth satellite is solved in this formulation by M. D. Kislik[1] in [69, 79], (sic).

According to these works, a system of curvilinear coordinates is selected in which any point M of space q_1, q_2, q_3 is given by the intersection of three surfaces: an ellipsoid of revolution, a hyperboloid of one sheet which is confocal with this ellipsoid and has semitransverse axis a_2, and the meridian plane passing through their common axis (the minor axis of the ellipsoid and the conjugate axis of the hyperboloid). The ellipsoid and hyperboloid are confocal with some biaxial ellipsoid (with eccentricity $e = \sqrt{a^2 - b^2/b}$) which is selected in such a way that the gravitational potential $\overline{V}$ of the homogeneous body bounded by its surface coincides with the gravitational potential of the terrestrial spheroid (see §5):

$$V = \frac{\mu}{r}\left[1 + \left(\frac{r_0}{r}\right)^2 c_{20} P_{20}(\sin\varphi)\right]. \tag{13.1}$$

In this case, coordinates q_1 and q_2 (coordinate q_3 is the spherical longitude of the meridian plane) are determined by the relationships

$$q_1 = \frac{1}{e^2}\left(1 + \frac{\nu_1}{b^2}\right); \quad \nu_1 = a_1^2 - a^2;$$

$$q_2 = \frac{1}{c^2}\left(1 + \frac{\nu_2}{b^2}\right); \quad \nu_2 = a_2^2 - a^2,$$

while the force function $\overline{V}$ which satsifies the conditions $\overline{V} - V = 0$ (V being given by equation (13.1)) is equal to

$$\left.\begin{aligned} \overline{V} &= \frac{\mu}{d}\frac{\sqrt{q_1}}{q_1 - q_2}; \\ d &= \sqrt{a^2 - b^2} = r_0\sqrt{|c_{20}|} \simeq 211 \text{ км}. \end{aligned}\right\} \tag{13.1'}$$

---

[1] A similar problem was solved somewhat later by J. Vinti [72].

However, the function $\overline{V}$ so selected coincides only with the first two terms of the force function of the biaxial terrestrial ellipsoid

$$V = \frac{\mu}{r}\left[1 + \sum_{i=1}^{\infty}\left(\frac{r_0}{r}\right)^{2i} c_{i0} P_{i0}(\sin\psi)\right].$$ /184

The resultant error is equal to

$$\Delta\overline{V} = \overline{V} - V = (\mu d^4 - c_{40})\frac{1}{r^5} P_{40}(\sin\psi) +$$
$$+ \frac{\mu}{r}\sum_{n=3}^{\infty}(-1)^n \frac{d^{2n}}{r^{2n}} P_{2n0}(\sin\psi).$$

Given certain assumptions, the first term of this expression alone may be used for evaluating error $\Delta\overline{V}$. According to M. D. Kislik's estimates, the error in determining the acceleration of a satellite when the potential is selected in the form of model (13.1') will be no greater than 7 mgal throughout all outer space.

In [69], potential (13.1) is called the normal gravitational field of the Earth. In this case, the quantity $\Delta\overline{V}$ appears in the potential of gravitational anomalies.

The equations are written in canonical form to find the integrals of the motion equations for the satellite in the normal gravitational field. Integration of the resultant system, following Yakobi's method [72] [sic], is replaced by finding the complete integral of some differential equation in partial derivatives. In this way, M. D. Kislik finds six integrals of canonical equations of motion which are expressed in quadratures, and after certain transformations may be written in the form:

$$\left.\begin{aligned}
q_3 &= q_{30} + \frac{D_2}{\sqrt{2}}[I_2(\eta) - I_1(\xi)];\\
I_3(\xi) &= I_4(\eta);\\
t &= \sqrt{\frac{d^3}{2\mu}}[I_5(\xi) + I_6(\eta)];\\
p_1 &= \sqrt{\frac{\mu d}{2}} \cdot \frac{\sqrt{\theta(\xi)}}{\xi(1+\xi^2)};\\
p_2 &= -\sqrt{\frac{\mu d}{2}}\,\frac{\sqrt{P(\eta)}}{\eta(1-\eta^2)};\\
p_3 &= D_2\sqrt{\mu d}.
\end{aligned}\right\} \qquad (13.2)$$

Here we use the notation: /185

$$\xi = \sqrt{q_1}; \quad \eta = \sqrt{-q_2};$$

$\theta(\xi)$ is a polynomial of the fourth degree;

I_1, I_2, I_3, I_4, I_5 and I_6 are elliptic integrals;

D_2 is a constant;

the subscript 0 corresponds to the initial values.

Actually, the derivation of integrals (13.2) constitutes construction of an analytical theory of satellite motion, called the hyperelliptic theory by M. D. Kislik; since the elliptic integrals written there are not transformed [73], equations (13.2) may not yet be used in computational practice. M. D. Kislik manages to reduce them to a form convenient for calculations by a certain transformation of integrals I_1, I_2, I_3, I_4, I_5 and I_6 and by expanding the integrands in series with respect to powers of the small parameters k_ξ^2, Δ_1, Δ_2 and β, which are approximately equal to

$$k_\xi^2 \simeq \varepsilon^2 e_0^2 \sin^2 i_0; \quad \Delta_1 \simeq 4\varepsilon^2 e_0 \cos^2 i_0;$$
$$\Delta_2 \simeq \varepsilon^2 e_0^2; \quad \beta \simeq \varepsilon^2 e_0 \cos 2i_0; \quad \varepsilon = \frac{d}{p_0}.$$

After these operations, the integrals are written in terms of the series $S(\varphi)$, $F(\varphi)$, $M(\varphi)$, $\Pi(\zeta)$ and $L(\zeta)$ as follows:

$$I_1 = A[S(\varphi) - S(\varphi_0)]; \quad I_4 = K[F(\zeta) - F(\zeta_0)];$$
$$I_2 = B[F(\zeta) - F(\zeta_0)]; \quad I_5 = N[M(\varphi) - M(\varphi_0)];$$
$$I_3 = C[F(\varphi) - F(\varphi_0)]; \quad I_6 = P[L(\zeta) - L(\zeta_0)],$$

where A, B, C, K, N and P are constants. The subscript 0 designates values at the initial point. The argument in these expressions is the angle ζ, which is approximately equal to the argument of latitude u in the corresponding Keplerian motion of the satellite. The exact value of ζ is found from the expression

$$\eta = \eta_2 \sin\zeta,$$

where $\eta_2 > 0$ is the lowest root of the polynomial $P(\eta) = -m\eta^4 + n\eta^2 - s$; m, n and s are constants. The angle φ is also expressed in terms of the angle ζ. In point of fact, however, (since $\eta = \sqrt{-q_2}$), it is obvious from (13.2) that the argument in this construction of the analytical theory is the curvilinear coordinate q_2. It should be noted that the independent variable may also be represented by the angular quantity φ (which is approximately equal to the true anomaly ϑ in Keplerian motion) or by the time of motion t. In this case /18
there is a change in the order of the calculations outlined below [70].

Let us assume that at any instant t_0, the given initial conditions are the spherical coordinates of the satellite r_0, ψ_0 and λ_0 and the corresponding components of velocity

$$v_{10} = \dot{r}_0; \quad v_{20} = r_0 \dot{\psi}_0; \quad v_{30} = r_0 \dot{\lambda}_0 \cos\psi_0.$$

Then the algorithm for determining satellite motion is constructed according to hyperelliptic theory as follows [9, 10].

1. The initial values of ξ_0, η_0, q_{10} and q_{20} are found:

$$\left.\begin{aligned}\xi_0 &= \frac{r_0}{d}\left\{1-\frac{1}{8}\frac{d^2}{r_0^2}\cos^2\psi_0\left[4+\frac{d^2}{r_0^2}(1-5\sin^2\psi_0)\right]\right\};\\ \eta_0 &= \sin\psi_0\left\{1+\frac{1}{8}\frac{d^2}{r_0^2}\cos^2\psi_0\left[4+\frac{d^2}{r_0^2}(3-7\sin^2\psi_0)\right]\right\};\end{aligned}\right\} \qquad (13.3)$$

$$\left.\begin{aligned}q_{10} &= \frac{1}{2}\left[\left(\frac{r_0^2}{d^2}-1\right)+\sqrt{\left(\frac{r_0^2}{d^2}-1\right)^2+4\frac{r_0^2}{d^2}\sin^2\psi_0}\right];\\ q_{20} &= \frac{1}{2}\left[\left(\frac{r_0^2}{d^2}-1\right)-\sqrt{\left(\frac{r_0^2}{d^2}-1\right)^2+4\frac{r_0^2}{d^2}\sin^2\psi_0}\right].\end{aligned}\right\} \qquad (13.4)$$

Here the quantity r_0 is the initial focal radius of the satellite's position; d is found from (13.1).

2. The values of the constants D_1, D_2, D_3, m, n and s are calculated:

$$D_1 = \frac{dv_0^2}{2\mu} - \frac{\sqrt{q_{10}}}{q_{10}-q_{20}};$$

$$D_2 = \frac{v_{30} r_0 \cos\psi_0}{\sqrt{\mu d}} = v_{30}\sqrt{\frac{d}{\mu}}\sqrt{(1+q_{10})(1+q_{20})};$$

$$D_3 = \frac{d}{4\mu}\left[\frac{2\mu}{d}\frac{q_{20}\sqrt{q_{10}}}{q_{10}-q_{20}} - (1+q_{10}+q_{20})(v_{20}^2+v_{30}^2) + {} \right.$$
$$\left. {} + (v_{10}\sin\psi_0 + v_{20}\cos\psi_0)^2\right];$$

$$m = D_1,\quad n = D_1 + 2D_3,\quad s = \frac{1}{2}D_2^2 + 2D_3,\quad v_0^2 = v_{10}^2 + v_{20}^2 + v_{30}^2.$$

The following approximate relationships may be used for a check (p_0 and e_0 /187 are the initial values of the focal parameter and eccentricity):

$$p_0 \simeq -2nd;\quad e_0 \simeq \sqrt{1-4mn}.$$

3. The method of successive approximations is used for finding the real roots ξ_1 and ξ_2 of the polynomial

$$\Theta(\xi) = m\xi^4 + \xi^3 + n\xi^2 + \xi + s; \quad \xi_1 \geq \xi_k \geq \xi_2 > 0.$$

The values

$$\xi'_1 = -\frac{2n}{1-\sqrt{1-4mn}}; \quad \xi'_2 = -\frac{2n}{1+\sqrt{1-4mn}}.$$

are taken as the first approximation. The following approximate relationships may be used for checking:

$$\xi_1 \simeq \frac{r_{max}}{d} = \frac{p_0}{d(1-e_0)}; \quad \xi_2 \simeq \frac{r_{min}}{d} = \frac{p_0}{d(1+e_0)}.$$

In the case of an orbit of a nearly circular satellite ($e \approx \varepsilon^2$), roots ξ_1 and ξ_2 differ from each other by a quantity approximately equal to 1. In this case, the values

$$\xi'_1 = \frac{p_0}{d} + \Delta\xi; \quad \xi'_2 = \frac{p_0}{d} - \Delta\xi; \quad \Delta\xi = 1 - 2.$$

are taken as the first approximation.

4. The quantities x_1 and x_2 are determined as the roots of the quadratic equation

$$(\alpha_1 - \alpha_2)x^2 + 2(\beta_1 - \beta_2)x + (\alpha_2\beta_1 - \alpha_1\beta_2) = 0;$$

$$\alpha_1 = -(\xi_1 + \xi_2); \qquad \beta_1 = \xi_1\xi_2;$$

$$\alpha_2 = \frac{1}{\beta_1}\left(\frac{1}{m} - \alpha_1\beta_2\right); \qquad \beta_2 = \frac{s}{m\xi_1\xi_2}.$$

5. The constants β, E, k^2_ξ, T_ξ, Δ_1, Δ_2 and Φ_0 are found:

$$\beta = \frac{x_2}{x_1} E; \qquad E = \sqrt{\frac{2x_1+\alpha_1}{2x_2+\alpha_1}};$$

$$k_\xi^2 = \frac{(2x_1+\alpha_1)(2x_2+\alpha_2)}{2(x_1-x_2)(\alpha_2-\alpha_1)}; \qquad T_\xi = \sqrt{\frac{2(\alpha_1+\alpha_2)}{(x_1-x_2)(\alpha_1-\alpha_2)}};$$

$$\Delta_1 = \frac{2E(1+x_1x_2)}{1+x_1^2}; \qquad \Delta_2 = \frac{E^2(1+x_2^2)}{1+x_1^2};$$

$$\varphi_0 = \begin{cases} \arccos\dfrac{x_1-\xi_0}{E(\xi_0-x_2)} & \text{where } \dot{q}_{10}>0; \\ 2\pi - \arccos\dfrac{x_1-\xi_0}{E(\xi_0-x_2)} & \text{where } \dot{q}_{10}<0. \end{cases}$$ /188

Derivative $\dot{q}_{10}$ is determined from the formula

$$\dot{q}_{10} = \frac{4q_{10}(1+q_{10})}{d^2(q_{10}-q_{20})} P_{20}; \quad P_{20} = \sqrt{\frac{\mu d}{2}} \frac{\sqrt{\theta(\xi_0)}}{\xi_0(1+\xi_0^2)}.$$

6. The quantities η_1 and η_2 are found as roots of the polynomial

$$P(\eta) = -m\eta^4 + n\eta^2 - s;$$
$$0 < \eta_2^2 \leq 1 < \eta_1^2.$$

To avoid loss of accuracy during the calculations, it is recommended that root η_2 be computed from the formula

$$\eta_2 = \sqrt{\frac{s}{m\eta_1^2}}.$$

7. The values of constants k_η^2, T_η and angle ζ_0 are calculated:

$$k_\eta^2 = \frac{\eta_2^2}{\eta_1^2}; \quad T_\eta = \frac{1}{\sqrt{-m\eta_1^2}} \quad (0 < k_\eta^2 < 1);$$

$$\zeta_0 = \begin{cases} \arcsin\frac{\eta_0}{\eta_2} & \text{where} \quad \frac{\dot q_{20}}{\eta_0} < 0; \\ \pi - \arcsin\frac{\eta_0}{\eta_2} & \text{where} \quad \frac{\dot q_{20}}{\eta_0} > 0; \end{cases}$$

$$\dot q_{20} = -\frac{4q_{20}(1+q_{20})}{d^2(q_{20}-q_{10})}\sqrt{\frac{b_0 d}{2}}\,\frac{\sqrt{P(\eta_0)}}{\eta_0(1-\eta_0^2)}.$$

If $\eta_0 = 0$ (the initial position of the satellite is in the plane of the equator), then the value $\zeta_0 = 0$ is taken at $v_{20} > 0$, and the value $\zeta_0 = \pi$ when $v_{20} < 0$.

8. The values of sums $S(\varphi_0)$, $F(\varphi_0)$, $M(\varphi_0)$, $\Pi(\zeta_0)$, $F(\zeta_0)$ and $L(\zeta_0)$ are calculated:

$$S(\varphi) = S_0' + 2ES_0'' + E^2 S_0''' + \sum_{n=1}^{\infty}\frac{1\cdot 3\ldots(2n-1)}{2\cdot 4\ldots 2n}\times$$
$$\times k_\xi^{2n}(S_n' + 2ES_n'' + E^2 S_n'''),$$

where /189

$$\left.\begin{array}{l} \left.\begin{array}{l} S_n' = \sum_{m=0}^{\infty} a_m D_{mn}(\varphi); \\ S_n'' = \sum_{m=0}^{\infty} a_m D_{m+1,n}(\varphi); \\ S_n''' = \sum_{m=0}^{\infty} a_m D_{m+2,n}(\varphi); \end{array}\right\} n = 0, 1, 2, \ldots \\ a_m = -\Delta_1 a_{m-1} - \Delta_2 a_{m-2}; \; m \geq 2; \; a_0 = 1; \; a_1 = -\Delta_1; \\ D_{mn} = \frac{2n-1}{2n+m} D_{m,n-1} - \frac{\sin^{2n-1}\varphi\cos^{m+1}\varphi}{2n+m}; \\ D_{mn} = \frac{m-1}{2n+m} D_{m-2,n} + \frac{\sin^{2n+1}\varphi\cos^{m-1}\varphi}{2n+m}; \\ D_{00}(\varphi) = \varphi; \quad D_{10}(\varphi) = \sin\varphi; \end{array}\right\} \quad (13.5)$$

$$\left.\begin{array}{l} F(\varphi) = A_0(\varphi) + \sum_{n=1}^{\infty}\frac{1\cdot 3\ldots(2n-1)}{2\cdot 4\ldots 2n} k_\xi^{2n} A_n(\varphi); \\ A_n(\varphi) = D_{0n}(\varphi); \end{array}\right\} \quad (13.6)$$

$$\left.\begin{aligned}
M(\varphi) &= M_0' + 2\beta M_0'' + \beta^2 M_0''' + \sum_{n=1}^{\infty} \frac{1\cdot 3\ldots(2n-1)}{2\cdot 4\ldots 2n} \times \\
&\qquad \times k_{\xi}^{2n} (M_n' + 2\beta M_n'' + \beta^2 M_n''');\\
M_n'(\varphi) &= \sum_{m=0}^{\infty} (-1)^m (m+1) E^m D_{mn};\\
M_n''(\varphi) &= \sum_{m=0}^{\infty} (-1)^m (m+1) E^m D_{m+1,n};\\
M_n'''(\varphi) &= \sum_{m=0}^{\infty} (-1)^m (m+1) E^m D_{m+2,n}.
\end{aligned}\right\} \qquad (13.7)$$

At large values of e_0, in view of the slow convergence of these series, the use of recurrence formulas is recommended: /190

$$M_n' = M_{n-1}' - M_{n-1}'';$$

$$M_n'' = \frac{1}{E^2}\left[(1+E^2) M_{n-1}'' + 2E M_{n-1}'' - \frac{\sin^{2n-1}\varphi}{2n-1}\right];$$

$$M_n''' = \frac{1}{E^2}\left[A_n - M_n' - 2E M_n''\right];$$

$$M_0' = -\frac{1}{1-E^2}\left[\frac{E\sin\varphi}{1+E\cos\varphi} - \frac{2}{\sqrt{1-E^2}}\sigma(\varphi)\right];$$

$$M_0'' = \frac{1}{1-E^2}\left[\frac{\sin\varphi}{1+E\cos\varphi} - \frac{2E}{\sqrt{1-E^2}}\sigma(\varphi)\right];$$

$$M_0''' = \frac{1}{E^2}\left[\varphi - \frac{E\sin\varphi}{(1-E^2)(1+E\cos\varphi)} + \frac{2(2E^2-1)}{(1-E^2)\sqrt{1-E^2}}\sigma(\varphi)\right];$$

$$\sigma(\varphi) = \operatorname{arctg}\frac{\sqrt{1-E^2}}{1+E}\operatorname{tg}\frac{\varphi}{2}.$$

The angle σ lies in the same quadrant as the angle $\varphi/2$.

Where

$$\left.\begin{aligned} \Pi(\zeta) &= H_0 + \sum_{n=1}^{\infty} \frac{1\cdot 3\ldots(2n-1)}{2\cdot 4\ldots 2n} k_\eta^{2n} H_n, \\ H_n &= -\frac{1}{\eta_2^2}[A_{n-1}(\zeta) - H_{n-1}], \quad n=1,2,\ldots; \\ H_0 &= \frac{1}{\sqrt{1-\eta_2^2}} \operatorname{arctg}(\sqrt{1-\eta_2^2}\, \operatorname{tg}\zeta) = \frac{1}{\sqrt{1-\eta_2^2}} \chi(\zeta). \end{aligned}\right\} \quad (13.8)$$

The angle $\chi(\zeta) = \arctan(\sqrt{1-\eta_2^2} \tan\zeta)$ lies in the same quadrant as the angle ζ.

The sum $F(\xi)$ is found from relationship (13.6), but with substitution of the variable ξ for the variable φ, and substitution of the constant k_η^2 for the constant k_ξ^2.

$$L(\zeta) = A_1(\zeta) + \sum_{n=1}^{\infty} \frac{1\cdot 3\ldots(2n-1)}{2\cdot 4\ldots 2n} k_\eta^{2n} A_{n+1}(\zeta). \quad (13.9)$$

In all groups of formulas (13.5), (13.6), (13.7), (13.8) and (13.9), the initial values φ_0 and ζ_0 are taken as arguments φ and ζ in calculating the sums $S(\varphi_0)$, $F(\varphi_0)$, $\Pi(\zeta_0)$, $F(\zeta_0)$ and $L(\zeta_0)$. /191

This completes determination of the constants and of the quantities which depend on the initial conditions of motion. Subsequent computations are done for each given instantaneous value of the argument ζ.

9. For each given value of ζ, the following precedure is used for calculating the corresponding value of φ:

-- $F(\zeta)$ is calculated from formula (13.6), the variable ζ being substituted for φ, and the constant k_η^2 being substituted for k_ξ^2;

-- the function $F(\varphi)$ is found from the equation

$$F(\varphi) = F(\varphi_0) + \frac{T_\eta}{T_\xi}[F(\zeta) - F(\zeta_0)];$$

-- the angle φ is found by the formula

$$\varphi = \frac{\pi F(\varphi)}{2K(k_\xi)} + 2\sum_{n=1}^{\infty}\frac{1}{n}\frac{q^n}{1+q^{2n}}\sin\left[\frac{n\pi F(\varphi)}{K(k_\xi)}\right],$$

in which the function $K(k_\xi)$ is the series

$$K(k_\xi) = \frac{\pi}{2}\left[1+2\sum_{n=1}^{\infty} q^{n^2}\right]^2,$$

$$q = \frac{1}{16}k_\xi^2\left(1+\frac{1}{2}k_\xi^2+\frac{21}{64}k_\xi^4+\frac{31}{128}k_\xi^6+\ldots\right).$$

Actually, the integral $I_3(\xi) = I_4(\eta)$ from equations (13.2) is used in the given case.

10. The instantaneous values are calculated for the integrals:

$$I_1 = \frac{T_\xi}{1+x_1^2}[S(\varphi) - S(\varphi_0)];$$

$$I_2 = T_\eta[\Pi(\zeta) - \Pi(\zeta_0)];$$

$$I_5 = T_\xi x_1^2[M(\varphi) - M(\varphi_0)];$$

$$I_6 = T_\eta \eta_2^2[L(\zeta) - L(\zeta_0)].$$

After this, q_3, t_1, p_1, p_2 and p_3 may be found according to relationships (13.2). Coordinates q_1 and q_2 (the latter, as has already been stated, is the argument) are determined as follows: /192

$$\xi = x_1\frac{1+\beta\cos\varphi}{1+E\cos\varphi};\quad \eta = \eta_2\sin\zeta;\quad q_1 = \xi^2;\quad q_2 = -\eta^2.$$

The constants β and E were found previously (see step 5).

Reverse transformation to coordinates and velocities (in the spherical coordinate system) is done by the formulas

$$r = d\sqrt{1+q_1+q_2};$$

$$\mathrm{tg}^2\psi = -\frac{q_1 q_2}{(1+q_1)(1+q_2)}$$

(the symbol ψ coincides with the symbol η):

$$\lambda = q_3;$$

$$v_1 = \frac{2}{r(q_1-q_2)}[q_1(1+q_1)p_1 - q_2(1+q_2)p_2];$$

$$v_2 = \frac{r\sin 2\psi}{d^2(q_2-q_1)}(p_2-p_1);$$

$$v_3 = \frac{p_3}{d\sqrt{(1+q_1)(1+q_2)}}.$$

This completes calculation of the coordinates and velocities at each instantaneous point of the trajectory.

§14. Solution of Equations of Disturbed Satellite Motion in Rectangular Coordinates

Equations of disturbed motion of a satellite in a rectangular coordinate system with regard to terms of the second negative order of magnitude with respect to polar flattening of the Earth were solved by A. A. Orlov in 1960 [76, 77]. These papers are a continuation of [75], where the same solution was found only with regard to terms of the first order with respect to flattening α. The ideas developed in these works were already present in [74] (1953).

The practical value of A. A. Orlov's papers (and particularly of the solutions published in [77] which will be taken up in this section) lies in the fact that the relationships which describe the motion of an artificial Earth satellite are derived without imposing any limitations on the initial orbital parameters. Thus, they are valid for any initial values of eccentricity and inclination. /193

The Earth is represented as an ellipsoid of revolution with a gravitational potential which contains even zonal harmonics:

$$V = \frac{\mu}{r}\left[1 + c_{20}\left(\frac{r_0}{r}\right)^2 P_{20} + c_{40}\left(\frac{r_0}{r}\right)^4 P_{40} + \dots\right]. \tag{14.1}$$

Since A. A. Orlov considers only second-order polar oblateness, (14.1) contains three terms corresponding to P_{00}, P_{20} and P_{40}.

The differential equations of disturbed motion

$$\frac{d^2x}{dt^2} = \frac{\partial v}{\partial x}; \quad \frac{d^2y}{dt^2} = \frac{\partial v}{\partial y}; \quad \frac{d^2z}{dt^2} = \frac{\partial v}{\partial z} \tag{14.2}$$

are written in the generally accepted geocentric inertial coordinate system (axes ox and oy lie in the equatorial plane of the Earth, oz is directed along the Earth's axis of rotation).

The problem is formulated as follows: to find the disturbed satellite motion described by system of equations (14.2) (assuming the form of function (14.1) given above) which would coincide with the Keplerian motion given at the initial instant when $c_{20} = c_{40} = 0$. The solution is found by transition to a new argument--undisturbed true anomaly ϑ_k given by the formula

$$\frac{d\vartheta_k}{dt} = \frac{\sqrt{\mu p_0}}{r_k^2}, \quad \text{where} \quad r_k = \frac{p_0}{1+e_0\cos\vartheta_k}.$$

Four transformations of the coordinate system are then performed to represent the motion of the ascending node and of the perigee of the disturbed orbit in trigonometric form.

1. Conversion from coordinates x, y, z to coordinates x*, y*, z* by rotation with respect to the oz axis through the angle

$$\Omega = \Omega_0 + \mu\vartheta_k; \quad \mu = \text{const};$$

$$\left.\begin{aligned} x &= x^*\cos\Omega - y^*\sin\Omega; \\ y &= x^*\sin\Omega + y^*\cos\Omega; \\ z &= z^*. \end{aligned}\right\}$$

2. Conversion to the ξ, η, ζ coordinate system by the transformation

$$\left.\begin{aligned} x^* &= \xi; \\ y^* &= \eta\cos i_0 - \zeta\sin i_0; \\ z^* &= \eta\sin i_0 + \zeta\cos i_0. \end{aligned}\right\}$$ /194

In this way the coordinate system ξ, η, ζ is tied to the plane of motion (in which the ξ- and η-axes lie) and rotates in absolute space at a variable velocity.

3. Transformation of coordinates ξ, η, ζ to coordinates ξ_1, η_1, ζ_1

$$\left.\begin{aligned} \xi &= \xi_1 \cos u - \eta_1 \sin u; \\ \eta &= \xi_1 \sin u + \eta_1 \cos u; \\ \zeta &= \zeta_1, \end{aligned}\right\}$$

where $u = \omega_0 + (1 + \lambda)\,\vartheta_k$; $\lambda = \text{const}$, represents a transition to a system rotating at a variable velocity with respect to axis $o\zeta_1$. As we see, the motion of system ξ_1, η_1, ζ_1 is complex.

A. A. Orlov does not undertake to find purely periodic motion; however, the constants μ and λ are chosen in the discussion which follows so as to eliminate certain nonperiodic terms in the solution (in order to simplify its form).

4. The final transformation, which is used in [16], is to change the scale of the coordinates

$$\varphi = \frac{\xi_1}{r_k}; \quad \psi = \frac{\eta_1}{r_k}; \quad \theta = \frac{\zeta_1}{r_k},$$

which varies together with r_k.

The solution for the equations satisfied by the functions φ, ψ, θ is sought in the form of series in sines and cosines of the angles which are multiples of u.

Substitution of these solutions in differential equations results in an infinite system of equations of relatively variable coefficients (associated with periodic functions); these coefficients are also sought in the form of series in powers of the small parameter α:

$$\alpha = r_0/p_0,$$

where r_0 is the equatorial radius of the Earth; p_0 is the focal parameter of the orbit.

In its final form (with regard to coefficients of the potential expansion c_{20}, c_{20}^2, c_{40}), the solution obtained by A. A. Orlov reduces to the following algorithm.

The argument for description of motion, as has already been pointed out, /195 is the undisturbed value of true anomaly ϑ_k. The coefficients associated with the functions of this variable in the formulas given below are constant which depend only on the initial conditions and should be computed beforehand. The quantities q_2, q_4 (A. A. Orlov's notation) are equal to

$$q_2 = |c_{20}| = 0.109808 \cdot 10^{-2}; \quad q_4 = c_{40} = -0.151259 \cdot 10^{-5}.$$

The initial parameters $\bar{p}_0$, $\bar{a}_0$, $\bar{e}_0$ which figure in the formulas given below are some constants (constants of integration) and differ from the corresponding Keplerian elements found for these same initial coordinates and velocities.

Relative to the possibilities of determining these parameters with respect to initial conditions, everything holds that is stated concerning this point at the end of §15. The simplest method in the given instance is calculation of the constants $\bar{p}_0$, $\bar{a}_0$, $\bar{e}_0$ by means of iteration using the procedure described in §15.

The subsequent operational order with A. A. Orlov's algorithm is as follows:

1. The constants $\bar{a}_0$, μ and λ are calculated:

$$\bar{a}_0 = \frac{\bar{p}_0}{1-\bar{e}_0^2};$$

$$\mu = -q_2 \frac{3}{2}\left(\frac{r_0}{\bar{p}_0}\right)^2 \cos^2 i_0 + q_2^2\left(\frac{r_0}{\bar{p}_0}\right)^4 \left[\left(-\frac{75}{32}+\frac{27}{32}\frac{\bar{p}_0}{\bar{a}_0}\right)\cos\bar{i}_0 + \right.$$
$$\left. + \left(\frac{51}{32}-\frac{15}{32}\frac{\bar{p}_0}{\bar{a}_0}\right)\cos^3\bar{i}_0\right] + q_4\left(\frac{r_0}{\bar{p}_0}\right)^4\left(-\frac{5}{4}+\frac{3}{4}\frac{\bar{p}_0}{\bar{a}_0}\right)\left(-\frac{45}{8}\cos\bar{i}_0+\frac{105}{8}\cos^3\bar{i}_0\right);$$

$$\lambda = q_2\left(\frac{r_0}{\bar{p}_0}\right)^2\left(-\frac{3}{4}+\frac{15}{4}\cos^2 i_0\right) + q_2^2\left(\frac{r_0}{\bar{p}_0}\right)^4 \left[\left(-\frac{201}{128}+\frac{75}{128}\frac{\bar{p}_0}{\bar{a}_0}\right) + \right.$$
$$\left. + \left(\frac{639}{64}-\frac{189}{64}\frac{\bar{p}_0}{\bar{a}_0}\right)\cos^2 i_0 + \left(-\frac{525}{128}+\frac{135}{128}\frac{\bar{p}_0}{\bar{a}_0}\right)\cos^4 i_0\right] +$$

$$+ q_4\left(\frac{r_0}{\overline{p}_0}\right)^4 \left[\left(\frac{7}{2} - \frac{3}{2}\frac{\overline{p}_0}{\overline{a}_0}\right)\left(\frac{45}{64} - \frac{225}{32}\cos^2 i_0 + \frac{525}{64}\cos^4 i_0\right) + \right.$$

$$\left. + \left(\frac{5}{2} - \frac{3}{2}\frac{\overline{p}_0}{\overline{a}_0}\right)\left(-\frac{45}{16}\cos^2 i_0 + \frac{105}{16}\cos^4 i_0\right)\right].$$

2. The variables M, r_k, $\frac{dr_k}{d}$, u and Ω are calculated for each given value of the argument ϑ_k: /196

$$M = E - e_0 \sin E; \quad \mathrm{tg}\frac{E}{2} = \sqrt{\frac{1-e_0}{1+e_0}}\,\mathrm{tg}\frac{\vartheta_k}{2};$$

$$r_k = \frac{p_0}{1+e_0\cos\vartheta_k}; \quad \frac{dr_k}{d\vartheta} = \frac{p_0 e_0 \sin\vartheta_k}{(1+e_0\cos\vartheta_k)^2},$$

$$u = \omega_0 + (1+\lambda)\vartheta_k; \quad \Omega = \Omega_0 + \mu\vartheta_k.$$

All subsequent computations are also done for the same values of ϑ_k.

3. R*, ω* and ζ* are found:

$$R^* = r_k + \left(\frac{r_0}{\overline{p}_0}\right)^2 R^{(2)}; \quad \omega^* = u + \left(\frac{r_0}{\overline{p}_0}\right)^2 \omega^{(2)}; \quad \zeta^* = \left(\frac{r_0}{\overline{p}_0}\right)^2 \zeta^{(2)};$$

$$R^{(2)} = -\frac{1}{2} q_2 \overline{p}_0 (\cos^2 i_0 - \sin^2 i_0 \cos^2 u);$$

$$\omega^{(2)} = q_2\left[(\cos^2 i_0 - \sin^2 i_0 \cos^2 u)\frac{\overline{p}_0}{r_k^2}\frac{dr_k}{d\vartheta} + \right.$$

$$\left. + \left(-\frac{3}{8} + \frac{1}{2}\frac{p_0}{r_k}\right)\sin^2 i_0 \sin 2u\right];$$

$$\zeta^{(2)} = q_2 r_k\left(2\frac{p_0}{r_k^2}\frac{dr_k}{d\vartheta}\cos u - \frac{p_0}{r_k}\sin u\right)\sin i_0 \cos i_0.$$

4. The values of the functions $R^{(4)}$, $\omega^{(4)}$ and $\zeta^{(4)}$ are calculated:

$$
\begin{aligned}
R^{(4)} = {} & \frac{1}{2} r_k (\omega^{(2)})^2 + r_k \Big\{ \Big[q_2^2 \left(-\frac{3}{8} + \frac{3}{8} \sin^2 i_0 + \frac{15}{64} \sin^4 i_0 \right) + \\
& + q_4 \left(-\frac{9}{8} + \frac{45}{8} \sin^2 i_0 - \frac{315}{64} \sin^4 i_0 \right) \Big] \Big[\frac{2}{3} \left(\frac{p_0}{a_0} \right)^{1/2} + \frac{p_0^2}{r_k^3} \frac{dr_k}{d\vartheta} (\vartheta_k - M) \Big] + \\
& + q_2^2 \Big[\left(\frac{1}{2} \frac{p_0}{a_0} - \frac{25}{16} \frac{p_0}{r_k} + \frac{9}{16} \frac{p_0^2}{r_k^2} \right) + \left(-\frac{1}{2} \frac{p_0}{a_0} + \frac{13}{16} \frac{p_0}{r_k} - \frac{9}{16} \frac{p_0^2}{r_k^2} \right) \sin^2 i_0 + \\
& + \left(-\frac{9}{256} + \frac{3}{16} \frac{p_0}{a_0} + \frac{157}{128} \frac{p_0}{r_k} - \frac{45}{128} \frac{p_0^2}{r_k^2} \right) \sin^4 i_0 \Big] + \\
& + q_4 \left(\frac{15}{8} - \frac{75}{8} \sin^2 i_0 + \frac{525}{64} \sin^4 i_0 \right) \Big[\left(-\frac{7}{6} + \frac{2}{5} \frac{p_0}{a_0} \right) \frac{p_0}{r_k} - \frac{1}{30} \frac{p_0^2}{r_k^2} + \frac{1}{5} \frac{p_0^3}{r_k^3} \Big] \Big\} +
\end{aligned}
$$

/197

$$
\begin{aligned}
& + r_k \Big\{ \Big[\left(-\frac{21}{16} q_2^2 + \frac{45}{16} q_4 \right) \sin^2 i_0 + \left(\frac{45}{32} q_2^2 - \frac{105}{32} q_4 \right) \sin^4 i_0 \Big] \frac{p_0^2}{r_k^3} \frac{dr_k}{d\vartheta} \vartheta_k + \\
& + q_2^2 \Big[\left(\frac{1}{2} \frac{p_0}{a_0} + \frac{45}{4} \frac{p_0}{r_k} - \frac{27}{2} \frac{p_0^2}{r_k^2} \right) \sin^2 i_0 + \left(-\frac{1}{4} \frac{p_0}{a_0} + \frac{9}{4} \frac{p_0}{r_k} \right) \sin^4 i_0 \Big] + \\
& + q_4 \left(\frac{3}{16} \sin^2 i_0 - \frac{7}{32} \sin^4 i_0 \right) \Big[\left(-\frac{67}{2} - 12 \frac{p_0}{a_0} \right) \frac{p_0}{r_k} + \frac{139}{2} \frac{p_0^2}{r_k^2} - 14 \frac{p_0^3}{r_k^3} \Big] \Big\} \cos 2u + \\
& + r_k \Big\{ \Big[\left(\frac{21}{16} q_2^2 - \frac{45}{16} q_4 \right) \sin^2 i_0 + \left(-\frac{45}{32} q_2^2 + \frac{105}{32} q_4 \right) \sin^4 i_0 \Big] \left(-\frac{p_0}{r_k} + \frac{p_0^2}{r_k^2} \right) \vartheta_k + \\
& + q_2^2 \Big[\left(\frac{3}{8} - \frac{459}{32} \frac{p_0}{r_k} \right) \sin^2 i_0 + \left(-\frac{9}{16} + \frac{67}{64} \frac{p_0}{r_k} \right) \sin^4 i_0 \Big] \frac{p_0}{r_k^2} \frac{dr_k}{d\vartheta} + \\
& + q_4 \left(\frac{3}{8} \sin^2 i_0 - \frac{7}{16} \sin^4 i_0 \right) \left(25 \frac{p_0}{r_k} - 8 \frac{p_0^2}{r_k^2} \right) \frac{p_0}{r_k^2} \frac{dr_k}{d\vartheta} \Big\} \sin 2u + \\
& + r^{(0)} \Big\{ q_2^2 \sin^4 i_0 \left(\frac{9}{256} + \frac{1}{16} \frac{p_0}{a_0} - \frac{33}{128} \frac{p_0}{r_k} + \frac{17}{128} \frac{p_0^2}{r_k^2} \right) + q_4 \sin^4 i_0 \Big[\Big(\frac{5}{128} + \\
& + \frac{3}{32} \frac{p_0}{a_0} \Big) \frac{p_0}{r_k} - \frac{41}{128} \frac{p_0^2}{r_k^2} + \frac{19}{64} \frac{p_0^3}{r_k^3} \Big] \Big\} \cos 4u + r_k \Big[q_2^2 \sin^4 i_0 \left(-\frac{3}{32} + \frac{41}{256} \frac{p_0}{r_k} \right) + \\
& + q_4 \sin^4 i_0 \left(-\frac{29}{256} \frac{p_0}{r_k} + \frac{1}{4} \frac{p_0^2}{r_k^2} \right) \Big] \frac{p_0}{r_k^2} \frac{dr_k}{d\vartheta} \sin 4u;
\end{aligned}
$$

$$\omega^{(4)} = -\frac{R^{(2)}\omega^{(2)}}{r_k} + \left[q_2^2\left(-\frac{3}{8}\sin^2 i_0 + \frac{15}{64}\sin^4 i_0\right) + q_4\left(-\frac{9}{8} + \frac{45}{8}\sin^2 i_0 -\right.\right.$$

$$\left.\left. - \frac{315}{64}\sin^4 i_0\right)\right]\frac{p_0^2}{r_k^2}(\vartheta_k - M) + q_2^2\left(\frac{25}{8} - \frac{25}{8}\sin^2 i_0 - \frac{49}{64}\sin^4 i_0\right)\frac{p_0}{r_k^2}\frac{dr_k}{d\vartheta} +$$

$$+ q_4\left(-\frac{15}{8} + \frac{75}{8}\sin^2 i_0 - \frac{525}{64}\sin^4 i_0\right)\left(-\frac{7}{3} + \frac{8}{15}\frac{p_0}{a_0} + \frac{2}{15}\frac{p_0^2}{r_k^2}\right)\frac{p_0}{r_k^2}\frac{dr_k}{d\vartheta} + \left\{\left[\left(\frac{21}{16}q_2^2 -\right.\right.\right.$$

$$\left.\left. - \frac{45}{16}q_4\right)\sin^2 i_0 + \left(-\frac{45}{32}q_2^2 + \frac{105}{32}\right)\sin^4 i_0\right]\left(-\frac{7}{2} - \frac{5}{2}\frac{p_0}{a_0} + 10\frac{p_0}{r_k} - 4\frac{p_0^2}{r_k^2}\right)\vartheta_k +$$

$$+ q_2^2\left[\left(-\frac{57}{32} - \frac{455}{32}\frac{p_0}{r_k}\right)\sin^2 i_0 + \left(\frac{405}{32} - \frac{11}{32}\frac{p_0}{r_k}\right)\sin^4 i_0\right]\frac{p_0}{r_k^2}\frac{dr_k}{d\vartheta} +$$ /198

$$\left. + q_4\left(\frac{3}{16}\sin^2 i_0 - \frac{7}{32}\sin^4 i_0\right)\left(15 + 16\frac{p_0}{a_0} - 28\frac{p_0}{r_k} + 20\frac{p_0^2}{r_k^2}\right)\frac{p_0}{r_k^2}\cdot\frac{dr_k}{d\vartheta}\right\} \times$$

$$\times \cos 2u + \left\{\left[\left(-\frac{21}{16}q_2^2 + \frac{45}{16}q_4\right)\sin^2 i_0 + \left(\frac{45}{32}q_2^2 - \frac{105}{32}q_4\right)\sin^4 i_0\right] \times\right.$$

$$\times\left(-6 + 4\frac{p_0}{r_k}\right)\frac{p_0}{r_k^2}\frac{dr_k}{d\vartheta}\cdot\vartheta_k + q_2^2\left[\left(-\frac{57}{16} - \frac{171}{16}\frac{p_0}{r_k} + \frac{461}{32}\frac{p_0}{r_k^2}\right)\sin^2 i_0 + \left(-\frac{1425}{128} +\right.\right.$$

$$\left.\left. + \frac{53}{128}\frac{p_0}{a_0} + \frac{21}{2}\frac{p_0}{r_k}\right)\sin^4 i_0\right] + q_4\left(\frac{3}{16}\sin^2 i_0 - \frac{7}{32}\sin^4 i_0\right)\left[-2 + 32\frac{p_0}{a_0} +\right.$$

$$\left.\left. + \left(-38 - 24\frac{p_0}{a_0}\right)\frac{p_0}{r_k} + 47\frac{p_0^2}{r_k^2} - 20\frac{p_0^3}{r_k^3}\right]\right\}\sin 2u + \left[q_2^2\sin^4 i_0\left(\frac{3}{64} - \frac{3}{32}\frac{p_0}{r_k}\right) +\right.$$

$$\left. + q_4\sin^4 i_0\left(-\frac{15}{64} - \frac{1}{8}\frac{p_0}{a_0} + \frac{25}{32}\frac{p_0}{r_k} - \frac{17}{32}\frac{p_0^2}{r_k^2}\right)\right]\frac{p_0}{r_k^2}\frac{dr_k}{d\vartheta}\cos 4u +$$

$$+ \left\{q_2\sin^4 i_0\left(\frac{9}{512} + \frac{13}{512}\frac{p_0}{a_0} - \frac{9}{64}\frac{p_0}{r_k} + \frac{21}{256}\frac{p_0^2}{r_k^2}\right) + q_4\sin^4 i_0\left[-\frac{35}{512} -\right.\right.$$

$$\left.\left. - \frac{145}{512}\frac{p_0}{a_0} + \left(\frac{25}{32} + \frac{3}{8}\frac{p_0}{a_0}\right)\frac{p_0}{r_k} - \frac{329}{256}\frac{p_0^2}{r_k^2} + \frac{9}{16}\frac{p_0^3}{r_k^3}\right]\right\}\sin 4u;$$

$$\zeta^{(4)} = r_k\left\{\left[\frac{21}{16}q_2^2 - \frac{45}{16}q_4 + \left(-\frac{135}{64}q_2^2 + \frac{315}{64}q_4\right)\sin^2 i_0\right] \times\right.$$

$$\times e_0^2 \sin i_0 \cos i_0 \cdot \vartheta_k \cos 2\vartheta + q_2^2 \sin i_0 \cos i_0\left[-\frac{131}{16} + \frac{27}{16}\frac{p_0}{r_k} + \left(\frac{953}{64} -\right.\right.$$

$$
-\frac{105}{64}\frac{p_0}{r_k}\Big)\sin^2 i_0\Big]\frac{p_0}{r_k^2}\frac{dr_k}{d\vartheta}+q_4\Big(\frac{15}{4}-\frac{105}{16}\sin^2 i_0\Big)\sin i_0\cos i_0\Big(\frac{21}{4}-\frac{16}{15}\frac{p_0}{a_0}+
$$

$$
+\frac{1}{12}\frac{p_0}{r_k}-\frac{4}{15}\frac{p_0^2}{r_k^2}\Big)\frac{p_0}{r_k^2}\frac{dr_k}{d\vartheta}\Big\}\cos u+r_k\Big\{\Big[\frac{21}{16}q_2^2-\frac{45}{16}q_4+\Big(-\frac{135}{64}q_2^2+\frac{315}{64}q_4\Big)\sin^2 i_0\Big]\times
$$

$$
\times e_0^2\sin i_0\cos i_0\cdot\vartheta_k\sin 2\vartheta_k+q_2^2\sin i_0\cos i_0\Big(7\frac{p_0}{r_k}-\frac{193}{16}\frac{p_0}{r_k}\sin^2 i_0\Big)+
$$

/199

$$
+q_4\Big(\frac{15}{4}-\frac{105}{16}\sin^2 i_0\Big)\sin i_0\cos i_0\Big[\Big(-4+\frac{4}{5}\frac{p_0}{a_0}\Big)\frac{p_0}{r_k}+\frac{2}{5}\frac{p_0^3}{r_k^3}\Big]\Big\}\sin u+
$$

$$
+r_k\sin^3 i_0\cos i_0\Big\{\Big(-\frac{45}{64}q_2^2+\frac{105}{64}q_4\Big)e_0^2\vartheta_k\cos 2\vartheta_k+\Big[q_2^2\Big(\frac{111}{64}+\frac{1}{64}\frac{p_0}{r_k}\Big)+
$$

$$
+q_4\Big(-\frac{67}{64}+\frac{p_0}{a_0}-\frac{253}{64}\frac{p_0}{r_k}+\frac{9}{4}\frac{p_0^2}{r_k^2}\Big)\Big]\frac{p_0}{r_k^2}\frac{dr_k}{d\vartheta}\Big\}\cos 3u+r_k\sin^3 i_0\cos i_0\times
$$

$$
\times\Big\{\Big(-\frac{45}{64}q_2^2+\frac{105}{64}q_4\Big)e_0^2\vartheta_k\sin 2\vartheta_k+q_2^2\Big(\frac{171}{128}+\frac{21}{128}\frac{p_0}{a_0}-\frac{33}{16}\frac{p_0}{r_k}\Big)+
$$

$$
+q_4\Big[-\frac{177}{128}+\frac{183}{128}\frac{p_0}{a_0}+\Big(-\frac{3}{2}-\frac{9}{4}\frac{p_0}{a_0}\Big)\frac{p_0}{r_k}+\frac{51}{8}\frac{p_0^2}{r_k^2}-\frac{17}{8}\frac{p_0^3}{r_k^3}\Big]\Big\}\sin 3u.
$$

5. The coordinates of satellite motion are found in the absolute geocentric system in the following order:

$$
\left.\begin{aligned}
R &= R^* + \left(\frac{r_0}{p_0}\right)^4 R^{(4)};\\
\omega &= \omega^* + \left(\frac{r_0}{p_0}\right)^4 \omega^{(4)};\\
\zeta &= \zeta^* + \left(\frac{r_0}{p_0}\right)^4 \zeta^{(4)};
\end{aligned}\right\}
$$

$$
\xi = R\cos\omega;\qquad \eta = R\sin\omega;\qquad \zeta = \zeta;
$$

$$
\left.\begin{aligned}
x^* &= \xi;\\
y^* &= \eta\cos i_0 - \zeta\sin i_0;\\
z^* &= \eta\sin i_0 + \zeta\cos i_0;
\end{aligned}\right\}
$$

$$
\Omega = \Omega_0 + \mu\vartheta_k;
$$

$$\left.\begin{aligned} x &= x^* \cos\Omega - y^* \sin\Omega; \\ y &= x^* \sin\Omega + y^* \cos\Omega; \\ z &= z^*. \end{aligned}\right\}$$

The algorithm allows for considerable simplifications if orders of oblateness above the first are disregarded (it is assumed that $q_4 = 0$, $q_2^2 = 0$), or if higher powers of e_0 are disregarded when the initial orbital eccentricities are low.

In particular, when $q_4 = 0$, $q_2^2 = 0$, the quantities x, y, z , as may be readily seen, are simply equal to $x = x^{(2)}$, $y = y^{(2)}$, $z = z^{(2)}$, and the entire algorithm for determining the instantaneous coordinates of motion is limited to the first four points described above. /200

§15. Utilizing the Model of Two Attracting Centers for Solving the Problem of Disturbed Satellite Motion

In this version of seeking the solution for equations of disturbed motion, consideration is given to the force function represented in the form

$$V = \frac{fm}{r}\left\{1 + \sum_{k=2}^{\infty} J_k \left(\frac{R_0}{r}\right)^k P_k\left(\frac{z}{r}\right)\right\}, \tag{15.1}$$

where f is the gravitational constant, m is mass, R_0 is the equatorial radius of the Earth, P_k are Legendre's polynomials, J_k are constants which characterize the figure of the Earth and r is the distance of the satellite from the Earth's center. The rectangular coordinate system with the origin at the Earth's center of gravity is selected so that axes x and y lie in the plane of the equator, the axis z is pointed toward the vernal equinox and the triplet x, y, z is right-handed.

Let us introduce the new constant

$$c = R_0 \sqrt{-J_2}$$

and examine the force function

$$U = \frac{fm}{r}\left\{1 + \sum_{k=1}^{\infty} (-1)^k \left(\frac{c}{r}\right)^{2k} P_{2k}\left(\frac{z}{r}\right)\right\}. \tag{15.2}$$

It is readily seen that the expressions for V and U from (15.1) and (15.2) have different terms beginning with k = 3. This means that the force functions V and U differ from one another by a quantity of the second negative order of magnitude with respect to flattening of the Earth. Therefore, if an analytical solution can be found for the problem of satellite motion in the field of a body with force function U, then it is obvious that this solution will be closer to the exact solution (corresponding to force function V) than for instance motion along a Keplerian ellipse (this motion is defined by the force function W = fm/r).

As shown in [78, 79, 80], the coefficients of the expansion from (15.2) are identically coincident with the coefficients of the expansion in Legendre's polynomials with respect to the argument z/r of the auxiliary function /201

$$\frac{fm}{r}\left(\frac{1}{r_1}+\frac{1}{r_2}\right).$$

Here

$$r_1=\sqrt{x^2+y^2+(z-ic)^2};\quad r_2=\sqrt{x^2+y^2+(z+ic)^2};\quad i=\sqrt{-1}.$$

And so we finally take

$$U=\frac{fm}{r}\left(\frac{1}{r_1}+\frac{1}{r_2}\right). \tag{15.3}$$

The equations of satellite motion in field (15.3) have the form

$$\frac{d^2x}{dt^2}=\frac{\partial U}{\partial x};\quad \frac{d^2y}{dt^2}=\frac{\partial U}{\partial y};\quad \frac{d^2z}{dt^2}=\frac{\partial U}{\partial z}. \tag{15.4}$$

System of equations (15.4) may be integrated in quadratures, its solution being expressed in terms of the elliptical functions of some intermediate variable. The relationship between this intermediate variable and the time of motion may also be established.

However, it is not always convenient to use the solution directly in elliptical functions. To simplify its use, we take advantage of the fact that the absolute values of the elliptical functions which represent the solution of system (15.4) are small. These absolute values are of order $\varepsilon = c/r^*$, where r^* is some mean radius of the satellite's orbit. Substituting

$c = R_0\sqrt{-J_2}$, , we find that

$$\varepsilon = \sqrt{-J_2}\,\frac{R_0}{r^*}.$$

Since r* is always greater than R_0, we have $\varepsilon < \sqrt{-J_2}$, and for the Earth, $\varepsilon < 1/30$. Thus, the resultant elliptical functions may be expanded in powers of the moduli. These series quickly converge for absolute values corresponding to flattening of the Earth.

The final results are given below with retention of terms to ε^4 inclusive, taken from [80]. The solution depends on six arbitrary constants:

$$a_0^*,\ e_0^*,\ i_0^*,\ \omega_0^*,\ \Omega_0^*,\ M_0^*, \tag{15.5}$$

which generally speaking may be expressed in terms of the initial values of the Keplerian elements. We note that all elements (15.5) become Keplerian at $\varepsilon = 0$, with the exception of the constant ω_0^*, which is expressed in this case in terms of the Keplerian value of the angular distance of the perigee (ω_0) in the form /202

$$\omega_0^* = \omega_0 - \pi/2.$$

The method for determining these elements from the Keplerian elements will be shown.

And so, if quantities (15.5) are known for the satellite at time t_0, then the coordinates of the satellite, x, y, z, and its velocities $\dot{x}$, $\dot{y}$, $\dot{z}$ may be determined at time t by successive application of the following formulas:

1. The constant parameters which characterize the orbit are given (these calculations are performed once for a given orbit):

$$s = \sin i_0^*;\quad p = a_0^*(1 - e_0^{*2});\quad \varepsilon = c/p;$$

$$\bar{e} = e_0^*[1 + \varepsilon^2(1 - e_0^{*2})(1 - 2s^2)];$$

$$n = \sqrt{\frac{fm}{a_0^{*3}}}\left\{1 - \frac{3}{2}\varepsilon^2(1 - e_0^{*2})(1 - s^2) + \frac{3}{8}\varepsilon^4(1 - e_0^{*2})(1 - s^2)\times\right.$$

$$\times [1+11s^2-e_0^{*2}(1-5s^2)] - \frac{\varepsilon^4}{16}(1-e_0^{*2})^{3/2}(24-96s^2+75s^4)\Big\};$$

$$\nu = \frac{\varepsilon^2}{4}(12-15s^2) + \frac{\varepsilon^4}{64}[288-1296s^2+1035s^4 -$$
$$- e_0^{*2}(144+288s^2-510s^4)];$$

$$\mu = -\frac{3}{2}\varepsilon^2\left[1+\frac{1}{8}\varepsilon^2(6-17s^2-24e_0^{*2}s^2)\right]\cos i_0^*;$$

$$\bar{\nu} = \nu/(1+\nu); \quad \bar{\mu} = \mu/(1+\nu).$$

2. For each given time t, we calculate

a) $$M = n(t-t_0)+M_0^*;$$

b) by the method of successive substitutions, we find E from the equation

$$E = M + \bar{e}\sin E,$$

E = M may be taken as a first approximation for E;

c) we find the quantity θ by using the equation /203

$$\mathrm{tg}\frac{\theta}{2} = \sqrt{\frac{1+\bar{e}}{1-\bar{e}}}\,\mathrm{tg}\frac{E}{2}.$$

The relationships

$$\left.\begin{aligned}\sin\theta &= \sqrt{1-\bar{e}^2}\sin E/(1-\bar{e}\cos E);\\ \cos\theta &= (\cos E-\bar{e})/(1-\bar{e}\cos E).\end{aligned}\right\} \quad (15.6)$$

may be used for determining the quadrant in which θ is located. We note that the angle θ makes as many turns as E. Therefore, proper determination of θ requires referring the value found from formulas (15.6) to the initial revolution and adding $[E/2\pi]\,2\pi$ to the result;

d) we determine $\omega, \delta u, u, v, \Omega, \xi, \Omega', \varphi, \xi', \phi, \dot{\Omega}', N$:

$$\omega = \nu\theta + \omega_0^* ;$$

$$\delta u = -\frac{1}{4}\varepsilon^2 s^2 (1+e_0^*\cos\theta)^2 \sin 2(\theta+\omega) - \varepsilon^2 e_0^* (2 - 3s^2)\sin\theta - -\frac{1}{8}\varepsilon^2 e_0^{*2}(8 - 11s^2)\sin 2\theta;$$

$$u = (1+\nu)\theta + \omega_0^* + \delta u; \quad v = (1-\bar{\nu})(u - \omega_0^*); \quad \Omega = \bar{\mu}u + \Omega_0^* - \bar{\mu}\omega_0^*;$$

$$\frac{p}{\xi} = 1 + \frac{1}{2}\varepsilon^2 e_0^{*2}(1-2s^2) + e_0^*\left(1 - \frac{1}{16}\varepsilon^2 e_0^{*2} s^2\right)\cos v - -\frac{1}{2}\varepsilon^2 e_0^{*2}(1-2s^2)\cos 2v + \frac{1}{16}\varepsilon^2 e_0^{*3} s^2 \cos 3v;$$

$$\Omega' = \Omega - 2\varepsilon^2 e_0^* \cos i_0^* \sin v - \frac{1}{4}\varepsilon^2 e_0^{*2}\cos i_0^* \sin 2v;$$

$$\varphi = u + \frac{\pi}{2} - \frac{1}{8}\varepsilon^2 s^2 (1 - e_0^{*2})\sin 2u;$$

$$\xi' = \frac{\xi^2}{p}\left\{e_0^*\sin v - \frac{\varepsilon^2 e_0^*}{16}[24 - 32s^2 - e_0^{*2}(8 - 5s^2)]\sin v - -\varepsilon^2 e_0^{*2}(1-2s^2)\sin 2v + \frac{3}{16}\varepsilon^2 e_0^{*3} s^2 \sin 3v\right\};$$

$$\dot{\varphi} = 1 + \frac{\varepsilon^2}{4}[6 - 7s^2 + e_0^{*2}(2 - s^2)] - \frac{\varepsilon^2}{4}(1 - e_0^{*2})s^2\cos 2u;$$

$$\dot{\Omega}' = -\frac{3}{2}\varepsilon^2\cos i_0^* - 2\varepsilon^2 e_0^*\cos i_0^*\cos v - \frac{1}{2}\varepsilon^2 e_0^{*2}\cos i_0^*\cos 2v;$$

$$N = \frac{\sqrt{fmp}}{\xi^2 + c^2 s^2 \sin^2\varphi};$$

e) we determine the satellite's coordinates x, y, z and velocity components $\dot{x}$, $\dot{y}$, $\dot{z}$: /204

$$x = \sqrt{\xi^2 + c^2}(\cos\varphi\cos\Omega' - \cos i_0^*\sin\varphi\sin\Omega');$$

$$y = \sqrt{\xi^2 + c^2}(\cos\varphi\sin\Omega' + \cos i_0^*\sin\varphi\cos\Omega');$$

$$z = \xi\sin i_0^*\sin\varphi;$$

$$\dot{x} = N\left\{\frac{x\xi\xi'}{\xi^2 + c^2} - \sqrt{\xi^2 + c^2}(\sin\varphi\cos\Omega' + \cos i_0^*\cos\varphi\sin\Omega')\dot{\varphi} - y\dot{\Omega}'\right\};$$

$$\dot{y} = N\left\{\frac{y\xi\xi'}{\xi^2 + c^2} + \sqrt{\xi^2 + c^2}(-\sin\varphi\sin\Omega' + \cos i_0^*\cos\varphi\cos\Omega')\dot{\varphi} + x\dot{\Omega}'\right\};$$

$$\dot{z} = N\left\{\frac{z\xi'}{\xi} + \xi\cos\varphi\sin i_0^*\dot{\varphi}\right\}.$$

Arbitrary constants (15.5) on which the general solution of the problem depends, are related in a complex way to the initial values of the coordinates and velocities of the satellite. For instance, let us assume that it is known that when $t = t_0$, the Keplerian elements of the orbit to be computed are (giving the Keplerian elements is equivalent to giving the phase coordinates and vice versa)

$$a_0,\ e_0,\ i_0,\ \omega_0,\ \Omega_0,\ M_0. \tag{15.7}$$

The values of parameters (15.5) must be determined with respect to elements (15.7).

The inverse problem is solved by the formulas presented in steps 1 and 2 of this section. Here, by giving the arbitrary values of parameters (15.5) when $t = t_0$, we determine the phase variables

$$x(t_0),\ y(t_0),\ z(t_0),\ \dot{x}(t_0),\ \dot{y}(t_0),\ \dot{z}(t_0). \tag{15.8}$$

If quantities (15.8) are known, the initial values of the Keplerian elements

$$a_0(t_0),\ e_0(t_0),\ i_0(t_0),\ \omega_0(t_0),\ \Omega_0(t_0),\ M_0(t_0).$$

may be determined.

However, we are always interested in the reverse. In fact, the initial values of the Keplerian orbital elements (15.7) are usually known for the orbit which must be computed, and it is necessary to use these elements in determining constants (15.5).

Our problem would be solved if we could reverse the formulas given above, i.e. if we could write relationships of the form

$$a = f(x,\ t_0), \tag{15.9}$$

where a is a vector from (15.5) and x is a vector in phase coordinate space. /205

This is rather difficult to accomplish, and besides the reversed formulas for determining the initial conditions will, generally speaking, be just as complex as the direct formulas used for predicting satellite motion.

We shall outline below the method proposed by I. A. Krylov and F. L. Chernous'ko for determining elements (15.5) with respect to (15.7) using only direct formulas. Since this method may be applied to other problems than the one treated in this section, we shall examine the procedure in general form.

Let us assume that we know the relationships

$$a_0 = f(a_0^*, \varepsilon). \tag{15.10}$$

Here a_0 and a_0^* are understood to indicate one-dimensional vectors, and ε is some small parameter. It is known that vectors a_0 and a_0^* differ little from one another, and when $\varepsilon = 0$,

$$a_0 = a_0^*. \tag{15.11}$$

From (15.10) and (15.11) we get

$$a_0 = f(a_0, 0).$$

Hence, it follows that

$$\left\| \frac{\partial f}{\partial a_0} \right\|_{\varepsilon=0} = E$$

(the existence and continuity of partial derivatives $\partial f/ \partial a_0^*$ is assumed at the given ε). Here, E is a unitary matrix. If it is assumed that partial derivatives $\partial f/\partial a_0^*$ are continuous with respect to ε, then we have

$$\left\| \frac{\partial f}{\partial a_0^*} \right\| \longrightarrow E \quad \text{where} \quad \varepsilon \to 0. \tag{15.12}$$

Let us consider the iteration process

$$a_0^{*(n+1)} = a_0 - f(a_0^{*(n)}, \varepsilon) + a_0^{*(n)} = h(a_0^{*(n)}, \varepsilon). \tag{15.13}$$

It may be assumed that

$$a_0^{*(1)} = a_0. \qquad (15.14)$$

Iteration process (15.13) with initial approximation (15.14) should converge, since it follows from (15.12) that all elements of the matrix of derivatives $\frac{\partial k(a_0^*, \varepsilon)}{\partial a_0^*} \to 0$ as $\varepsilon \to 0$, and initial approximation (15.14) is fairly close /206 to the solution of equation (15.10). Let us remember that the formulas

$$a_0 = f(a_0^*, \varepsilon)$$

are our initial direct formulas.

And so, assuming $a_0^{*(1)} = a_0$, we get

$$a_0^{*(2)} = a_0 - f(a_0, \varepsilon) + a_0;$$
$$a_0^{*(3)} = a_0 - f(a_0^{*(2)}, \varepsilon) + a_0^{*(2)}$$

The iteration process may be concluded when $|a_0^{*(n)} - a_0^{*(n-1)}|$ is less than the appropriately small number $\varepsilon_1 > 0$ which we select. The described iteration method was used in the computations.

When the given method is used for determining parameters (15.5) which appear in the formulas of §15, it must be remembered that the direct formulas determine the values of the phase variables, x, y, z, $\dot{x}$, $\dot{y}$, $\dot{z}$. Therefore, when these variables have been found from the direct formulas, they must be used to compute the corresponding values of the Keplerian elements. In addition, when $\varepsilon = 0$, the parameter ω_0^* is related to ω_0 by the expression

$$\omega_0^* = \omega_0 - \pi/2.$$

In the case where the Keplerian elements cannot be determined from the initial values of the phase coordinates, the iteration process naturally will not converge. For instance, if the initial phase coordinates from (15.8) correspond to a circular orbit, then a singularity appears in the formulas for determining the initial angular position of the perigee. Therefore, the iteration process cannot converge.

Let us note that Ye. P. Aksenov [81] [sic] reversed the formulas given in steps 1 and 2 of this section and got relationships (15.9). Thus, when quantities (15.8) are known, (15.9) may be used to determine elements (15.5). In this case, the value of the parameter n from step 1 may be determined with high precision from the initial coordinates and velocities of the satellite (15.8).

As has been shown by calculations made at the Shternberg State Astronomical Institute, using reversed formulas and the exact value of n gives considerably greater precision to the approximate formulas in steps 1 and 2.

§16. Comparative Analysis /207

A practically usable algorithn which describes the motion of an artificial Earth satellite, should satisfy the following basic requirements:

--optimum correspondence (in the sense of selecting the coordinate system, computational argument and make-up of the input information) to the principal problem to be solved and adaptability to related problems;

--its systematic error should be no greater than the permissible error;

--it should be fairly easily carried out by available means of computation;

--it should permit modification to include additional disturbing factors introduced through the improvement of practical techniques (for instance the dissipative effect of the atmosphere as the strongest factor which influences the trajectories of low-orbit satellites).

The first of these requirements can be considered only in each specific case. For instance, where total information on motion is required (i.e. all seven motion parameters), the method described in §15 which provides for direct computation (without numerical integration) of the coordinates alone cannot be used. If the specific nature of the given problem or the computational process makes it difficult to use rapidly changing functions, then osculating elements are the most convenient system of parameters. This is exemplified to a certain extent by the solution of problems in prediction given in §17. It is also shown in this section how the appropriate selection of a system of parameters facilitates the solution of additional problems associated with the need for predicting the motion of artificial Earth satellites.

The last requirement given above for algorithms stems from the unavoidable process of increasing complication in practical problems and improvement of the technical systems in which these algorithms are used. Specifically, it may be assumed that the permissible systematic error (i.e. the maximum value of the systematic error which permits solving the given problem) will decrease in the course of time in many engineering problems, and more and more rigid demands will be imposed on the accuracy of describing satellite motion. This requires accounting for effects of the second and higher negative orders of magnitude in the future[1]. In the algorithms described in §§12, 14 and 15, this problem has already been solved. Apparently , the hyperelliptic theory (§13) may also be used as a basis for constructing a theory of motion which reflects the effects of higher harmonics in the potential expansion. In principle, the effect of these harmonics may also be accounted for by the method of expansion in powers of a small parameter (§11). Of course, as may be seen from the examples in §§12, 14 and 15, these additions entail complication of the algorithms. /208

In addition to accounting for quantities of higher negative orders of magnitude in the gravitational effect of the Earth, it becomes necessary to account for other effects commensurate with them (they are enumerated in §1). Chief among these is the effect of the atmosphere in the case of satellites with a comparatively low perigee (in this case, this effect may have a quite appreciable magnitude). The introduction of factors of this type into the problem may make it necessary to discard the given algorithm and work out a new one. In some cases, however, (where the added effect is not great and allows using a linear theory, as for instance, in considering the effect of the atmosphere), an attempt may be made to take additional account of the new effect without reconstructing the available algorithm. This is obviously most simply accomplished when the algorithms described in §§11 and 12 are used.

Of decisive importance in evaluating an algorithm are the simplicity with which it is carried out and the magnitude of the systematic error. Since all algorithms represent a more or less approximate solution of the Cauchy problem the magnitude of the systematic error is influenced by two factors: the accuracy of the mathematical model of motion and the error inherent in the method used for solving this problem.

The model of motion in the given case (where the disturbing effect of the Earth's eccentricity alone is considered) is determined by the model for the potential of terrestrial attraction. In the case taken up by A. A. Orlov (§14), the Earth is represented in the form of an ellipsoid of revolution with regard to the square of polar flattening (α^2). Accounting for the second power of α yields appreciable results only for a protracted interval of satellite motion (see Chapter Two). In the hyperelliptic theory (§13), only a certain part of the effect of α^2 is taken into account, while in the method of expansion in powers of a small parameter (§11) it is completely disregarded. Since the effect of the square of oblateness (particularly /209

[1] Of course this applies primarily to long intervals of time of motion of the satellite.

short time interval[1]) is not great, all methods examined in this chapter (with the exception of the asymptotic method described in §12) are approximately on an equal level from the standpoint of accuracy of the mathematical model of motion. The errors introduced into calculations by this model may be determined on the basis of the first and second chapters.

The model of satellite motion found by Yu. G. Yevtushenko by the asymptotic averaging method (§12) is essentially different. This model is more accurate than those in the other algorithms described here. The approximate solution given in §12 makes it possible to calculate the evolution of satellite motion fairly readily and with a high degree of precision over a considerable interval of motion; the algorithm is especially simple in the case of nearly circular satellite orbits. It is important that the approximate solution makes it possible to account for the effect of tesseral harmonics.

The error inherent in the method is due to the suppositions and assumptions made in solving the Cauchy problem. All the algorithms described in this chapter give an approximate solution of the problem, since even when the solution is determined within the framework of the hyperelliptic theory, a numerical result can only be obtained by use of expansion in series with respect to powers of the small parameter.

The most effective procedure for evaluating the error inherent in the method is numerical comparison of the solution obtained by some theory with the results of numerical integration of differential equations of motion.

The system of differential equations should preferably be written in the same system of parameters in which the solution is found; the mathematical model of motion in both cases should naturally also be the same. /210

In this type of comparison, it is important to have a certain generalized error, or more precisely the modulus of this error which shows the total deviation of the calculated motion due to the effect which the methodological error has on all the motion parameters to be determined by the algorithm. Therefore, it is convenient to characterize the error inherent in the method by the moduli of vector differences:

---

[1] It is the interval of time over which motion is considered which determines the advisability of accounting for such disturbing gravitational effects as the square of polar flattening of the Earth. Actually, over short intervals, this effect is so small that it should not be taken into account. Over appreciable intervals, other effects such as the atmosphere must be considered also. At great distances from the Earth, where the atmosphere has little effect, the influence of the square of flattening will also be low, even over a long interval of time.

$$\Delta r = |\vec{r}_И - \vec{r}_A|, \quad \Delta v = |\vec{v}_И - \vec{v}_A|. \tag{16.1}$$

Here the subscript И designates a quantity determined by integrating the differential equations on a digital computer, while the subscript A designates a quantity obtained by calculation according to the analytical relationships of the algorithm being considered. The focal radii $\vec{r} = \vec{r}\,(x,y,z)$ depend only on the coordinates, while the velocities $\vec{v} = \vec{v}\,(\dot{x},\dot{y},\dot{z})$ depend only on the derivatives.

In view of the nonlinearity of the system of equations which describe satellite motion, the systematic error must be evaluated by calculating differences of form (16.1) at various points in the proposed range of initial conditions.

By way of example, graphs are given below for the change in systematic errors of some algorithms as a function of the interval of solution of the Cauchy problem.

Shown in Figures 107-109 is the change in errors Δr for the algorithm described in §11(the corresponding variants of initial conditions are given in Table 35). The function Δr in the given case is that used in §6 (6.10), and the error inherent in the method is characterized by the tube shown in Figure 3.

TABLE 35*

Variant Number	Initial Conditions						
	Ω_0	i_0	ω_0	e_0	h_A, km	h_Π, km	u_0
1	2	3	4	5	6	7	8
1	0	45°	–	0	300	300	0
2	0	45°	–	0	1000	1000	0
3	0	45°	0	0,0499	1000	300	0
4	0	45°	0	0,7352	36000	300	0
5	0	63°,4	0	0,0499	1000	300	0
6	0	0°	0	0,0499	1000	300	0
7	0	5°	0	0,0499	1000	300	0

* Tr. Note: Commas indicate decimal points.

TABLE 35 (Continued) *

1	2	3	4	5	6	7	8
8	0	45°	45°	0,0499	1000	300	45°
9	0	90°	0	0,0499	1000	300	0
10	0	45°	90°	0,0499	1000	300	0
11	0	45°	45°	0,0499	1000	300	0
12	0	45°	0°	0,8670	$4 \cdot 10^5$	300	0
13	0	89°	0°	0,0499	1000	300	0
14	0	1°	0°	0,0499	1000	300	0
15	0	45°	135°	0,0499	1000	300	0

Figs. 110 - 113 show the change in errors $\Delta r = |\vec{r}_{\Pi} - \vec{r}_A|$ and $\Delta r_{\Pi} = |\vec{r}_A - \vec{r}_{\Pi}|$ for the algorithm described in §14. Here $\vec{r}_A$ is the position of the satellite determined from the abbreviated relationships in step 6 §14, while $\vec{r}_{\Pi}$ is the same quantity calculated from the complete relationships in §14.

The errors of Ye. P. Aksenov's method (§15) are characterized by the quantity $\Delta r(t)$, graphs of which are shown in Figs. 114 and 115.

To determine the error in the averaging method of §12, satellite motions were computed by numerical integration of system (12.5) on a digital computer for the following four cases

1) $i = 45°$, $\Omega = L = q = k = 0$, $p = 6683{,}553$ km;
2) $i = 45°$, $\Omega = L = k = 0$, $q = 0{,}0499033799$; $p = 6996{,}08681$ km;
3) $i = 45°$, $\Omega = L = k = 0$, $q = 0{,}21692838$, $p = 8126{,}84243$ km;
4) $i = 45°$, $\Omega = L = k = 0$, $q = 0{,}7281693$, $p = 11515{,}7477$ km.

For the first three examples, 100 revolutions of the satellite around the Earth were calculated, while 40 revolutions were taken for the fourth variant. The second and fourth zonal harmonics were taken into consideration in the gravitational potential. Given in Table 36 are the results of computations of the evolution of satellite motion according to the approximate formulas in §12 and the numerically obtained exact solutions (N is the number

*Tr. Note: Commas indicate decimal points.

TABLE 36*

Variant Number	Solution	Ω	i	p, km	q	k	$L-360° N$	t, Hours	Δr, km
1	Exact	38°20′32″,88	45°	6 663,5542	0,000241	−0,000663	−4′29″	149,978314	0,27715
	Appropriate	38°20′29″,2	45°	6 663,5526	0,000241	−0,000661	−4′43″		
2	Exact	34°46′38″,5	44°59′58″,9	6 996,0096	0,040144	0,029337	3°15′46″	161,967653	0,41809
	Appropriate	34°46′35″,6	44°59′58″,9	6 996,0088	0,040135	0,029336	3°15′32″		
3	Exact	25°45′41″,2	44°59′56″,52	8 126,5691	0,192876	0,099085	9°52′37″	217,198162	0,62300
	Appropriate	25°45′39″,6	44°59′56″,54	8 126,5699	0,192873	0,099081	9°52′23″		
4	Exact	5°07′47″,22	44°59′59″,44	11 515,687	0,724892	0,068956	4°50′50″	422,05249	7,05845
	Appropriate	5°07′46″,94	44°59′59″,30	11 515,670	0,724898	0,068809	4°50′37″		

* Tr. Note: Commas indicate decimal points.

of revolutions of the satellite around the Earth). The dimensional time of motion is indicated in the next to last column of the table, and the error Δr in determining the radius-vector of the satellite is given in the last column.

The approximate formulas approach the exact solutions with a high degree of accuracy. The precision is especially high in the case of nearly circular orbits. The error in determining the position of a satellite after 100 revolutions around the Earth is 277 meters. As the orbital eccentricity increases, accuracy falls off, and for $e_0 \approx 0.2$, the error in satellite position is $\approx$7 km after only 40 revolutions. The error in determining the focal parameter is no greater than 20 meters in any case. The error in determination of the angular elements Ω, i and L is less than 20".

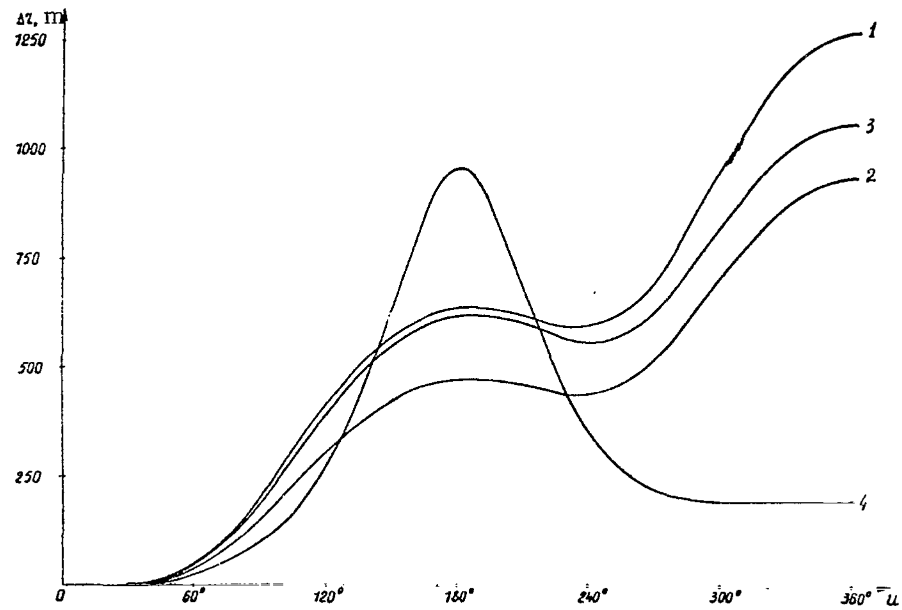


Fig. 107

/212

The important characteristics which determine the practical value of the algorithm are the size of the systematic error, the number of elementary operations and the storage volume necessary for carrying out the computation on a digital computer.

These data for the algorithms considered above are summarized in Table 37 (the systematic error in the given case is what was previously called the error inherent in the method). Indicated in this table are the

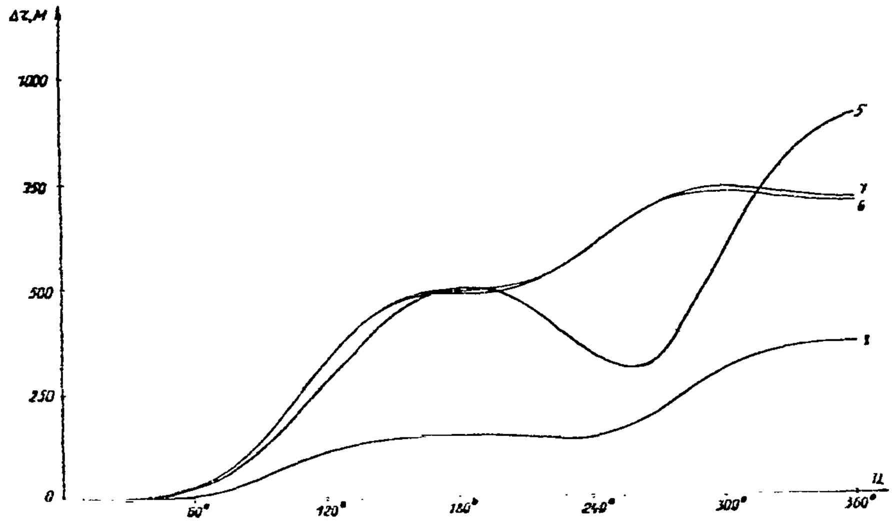


Fig. 108

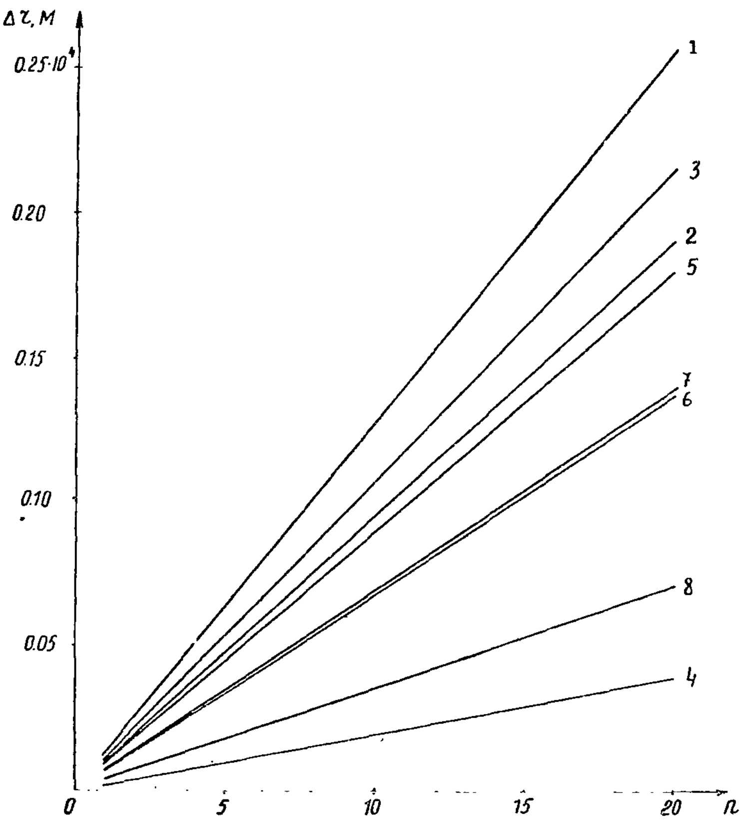


Fig. 109

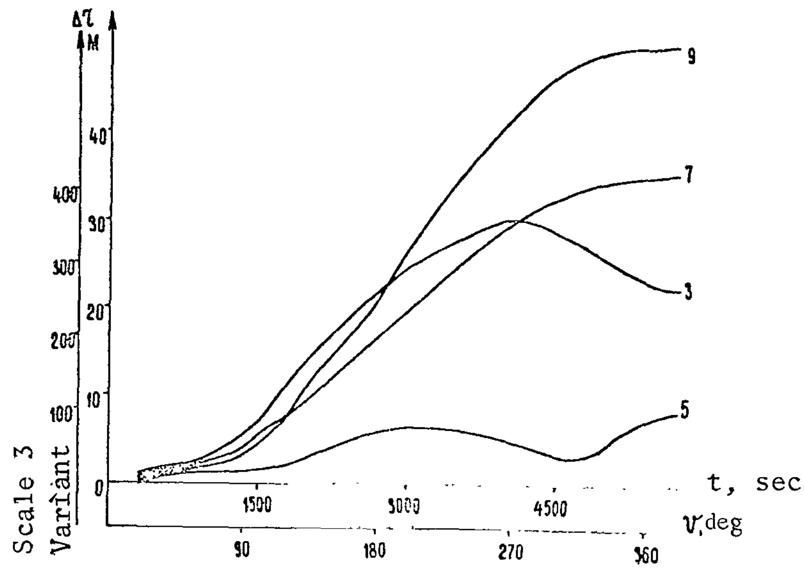


Fig. 110

/214

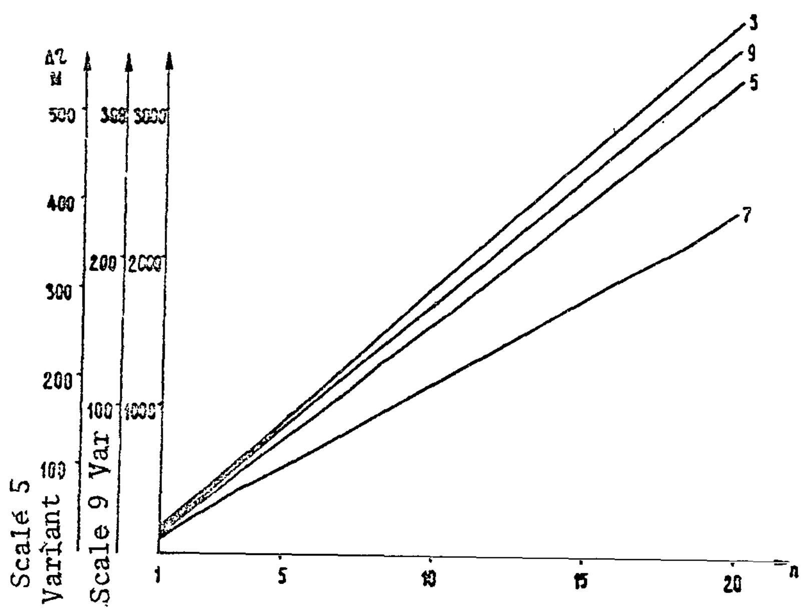


Fig. 111

maximum systematic errors obtained with a change in parameters over the ranges

$$0 < i_0 \leq 90°,\ 0 \leq \omega_0 \leq 90°,\ 300\ \text{km} \leq h_A \leq 36\ 000\ \text{km},\ 300\ \text{km} \leq h_\Pi \leq 1000\ \text{km}$$

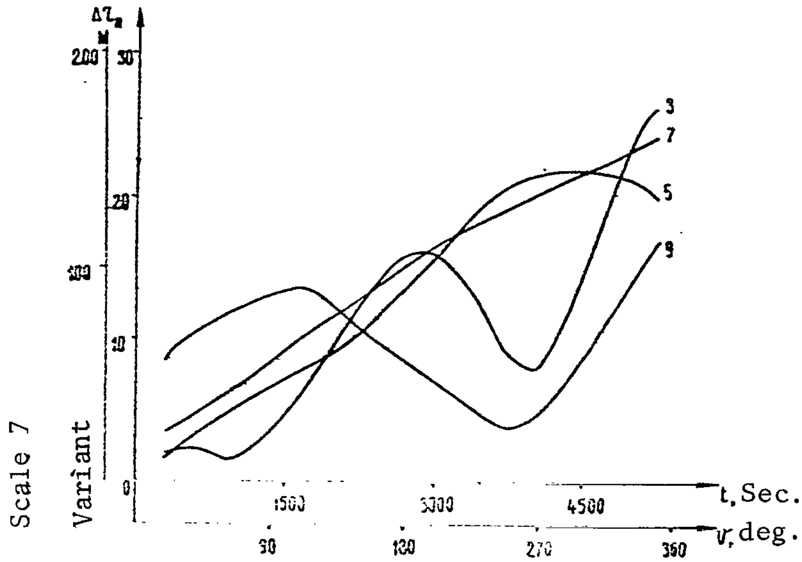


Fig. 112

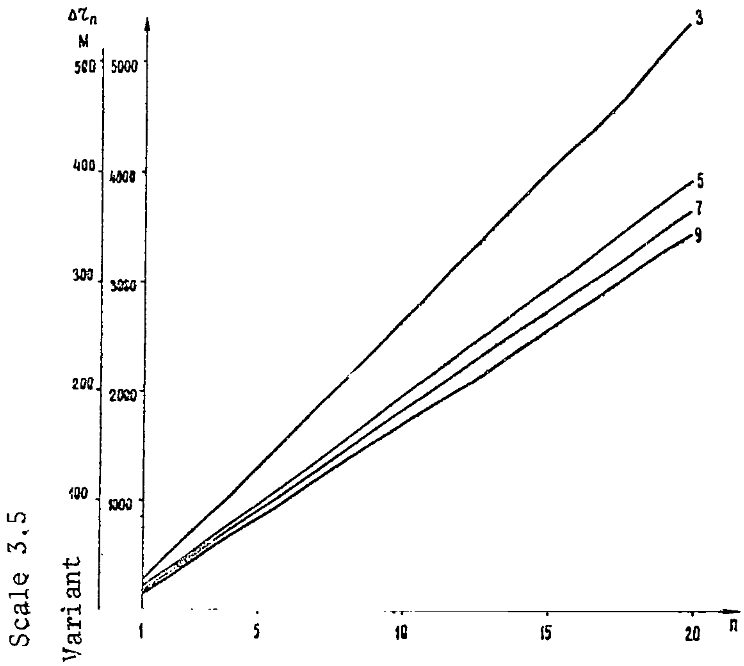


Fig. 113

(which corresponds to $6663.55\text{ km} \leq p_0 \leq 11\ 605.23\text{ km}$, $0 \leq e_0 \leq 0.7352$).
In cases where the algorithm permits accounting for both first and second powers of flattening, figures are given which characterize each of these

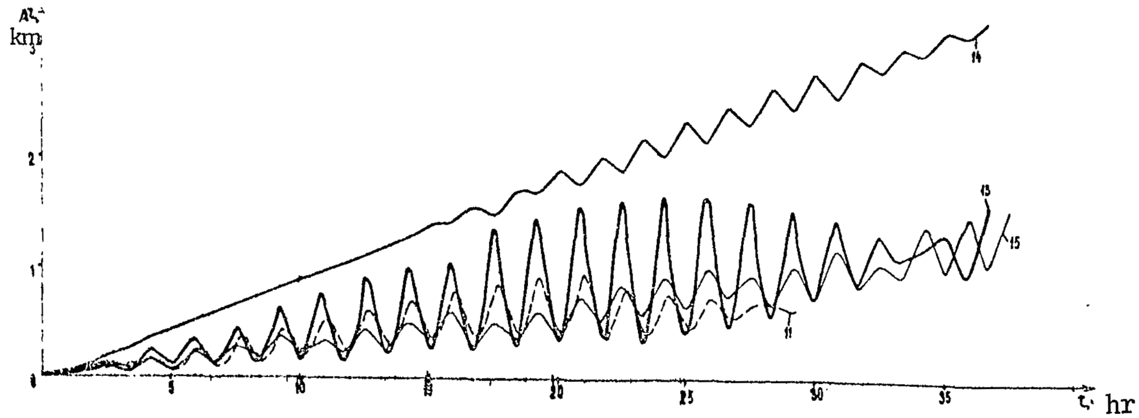


Fig. 114

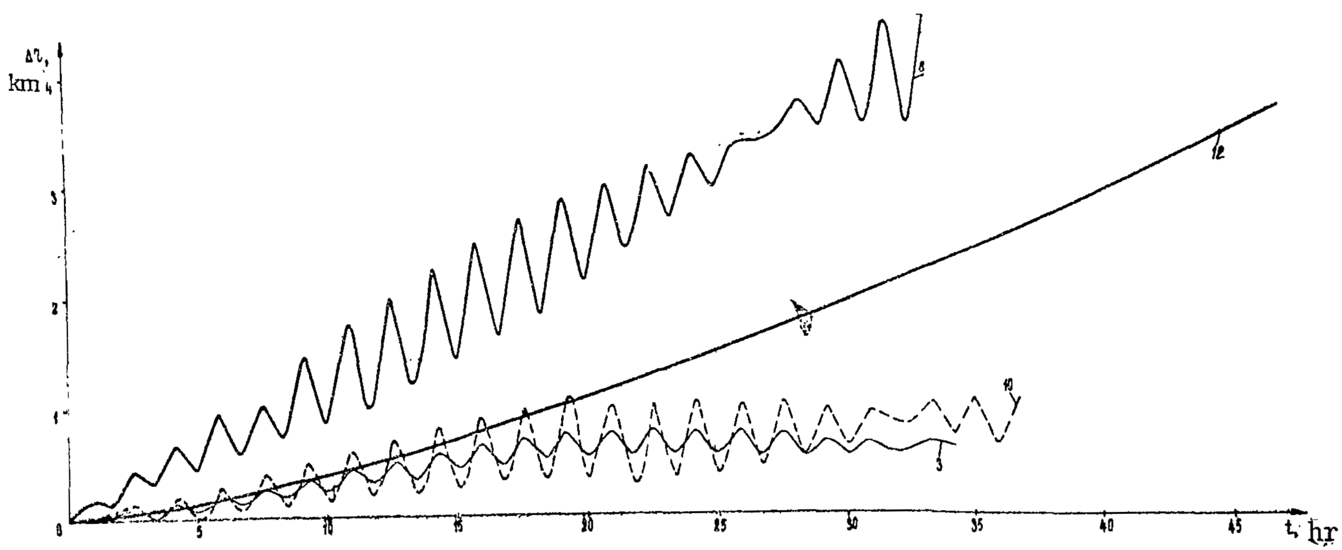


Fig. 115

TABLE 37 *

	Algorithm No. 11		Algorithm No. 12	Algorithm No. 13	Algorithm No. 14		Algorithm No. 15
	Complete Algorithm	Quantities of Order $O(e^2)$ Are Disregarded	For the Range $0 \leq e_0 \leq 0,2$ Over an Interval of motion of Less than 6 Days	Complete Algorithm	Δr	Complete Algorithm	Complete Algorithm
Systematic Errors on an Interval of: one revolution	$\Delta r \leq 1,3$ km $\delta t \leq 0,1$ sec	$\Delta r < 1,6$ km $\delta t < 0,12$ sec	$\Delta r \leq 1$ km	$\Delta r < 0,1$ km	$< 0,4$ km	$< 0,2$ km	$< 0,2$ km
one day	$\Delta r < 2,5$ km $\delta t < 0,3$ sec			$\Delta r < 0,2$ km	$< 3,2$ km	$< 3,5$ km	< 2 km
No. of Operations	250	220	1000	3000	400	–	700
Number of Numbers in the Computer	280	200	300	700	100	–	220

Note: The number of the algorithm corresponds to the number of the section in which it is described.

Tr. Note: Commas indicate decimal points.

variants separately. Also, in a separate column are the figures correspond- /219
ing to abbreviated algorithms (where such abbreviation is possible) in which quantities proportional to powers of eccentricity above the first are omitted.

In calculating the necessary number of operations and the required storage volume, no consideration was given to the calculations involved in converting the input quantities to the system of parameters which is used in a given algorithm or to those associated with reverse conversion of the quantities determined according to the algorithm into some other system of coordinates or parameters. It should be borne in mind that in certain instances (for example in the algorithm from §§14 and 15) the machine time required for carrying out these computations may be nearly the same as that for carrying out calculations by the algorithm itself.

"In studying the sciences, examples are no less instructive than rules."

I. Newton

Chapter Four

SOME EXAMPLES OF APPLICATION OF THE ANALYTICAL THEORIES OF MOTION OF AN ARTIFICIAL EARTH SATELLITE

The analytical theories of motion of an artificial Earth satellite may be most extensively used in solving various practical problems associated with the investigation of space and space flights.

Possible solutions for three such problems are given by way of example in this chapter. Of course in this regard, any analytical theory may be used (the one given in the Third Chapter or any other); however, the system selected in §17 has certain advantages for solving the problem of prediction (and related problems).

§17. Predicting the Motion of Artificial Earth Satellites

The problem of predicting satellite motion arises in connection with planning the trajectories of spacecraft [3], in connection with tracking a satellite from ground-based observation points for purposes of research or control, recognizing (identifying) it among a host of others of no interest at the given time, and in other similar problems.

Let us formulate the problem of prediction in a given instance. Let us assume we are given the set of parameters $F_{0i} = \{t_0, x_1^0, \ldots, x_N^0\}$ which characterize motion of a space vehicle when one of the variables $t = t_0$ is known. /221
Application of the operator S_{ti} transforms F_{0i} to the set $F_{ti} = \{t, x_1(t), \ldots, \ldots, x_N(t)\}$, which characterizes motion of the vehicle at any value of t:

$$F_{ti} = S_{ti} F_{0i}. \qquad (17.1)$$

In this regard all F_{ti} (accordingly F_{0i}) are elements of the set F_t

(accordingly of F_0): $F_t = \bigcup_i F_{ti}$, $F_0 = \bigcup_i F_{0i}$. Thus, we also have $S_{ti} \in S_t = \bigcup_i S_{ti}$.

Let us call the process of applying operator S_{ti} prediction of motion, the variable t we shall call the argument of prediction, and the interval of variation of the argument (in the given case $t - t_0$) will be the interval of prediction.

It is assumed that the set F_0 is given in the case of prediction, and the problem reduces to merely determining the specific form of the operator S_{ti}, although the manner of recording this operator depends on the type of selected subset of F_0. The selection of F_{ti} in particular, dictates a more or less complicated formal notation for the operator of prediction.

The motion of the center of gravity of the satellite is determined by six independent parameters given at a known moment of time. The general form of the operator of prediction S_{ti} in this case, is a system of nonlinear differential equations of the sixth order. Prediction actually reduces to solving the Cauchy problem, and F_{0i} represents the set of initial conditions. This system of equations, as has already been pointed out above, may be solved only by numerical or other approximate methods. Thus, the use of the prediction operator unavoidably introduces systematic errors into the quantities F_t.

In finding the form of S_t, we should strive to satisfy the requirements formulated in §16 and, in particular, the following two conditions:

-- the errors in determining F_t should not exceed the permissible value;

-- the formula for the prediction operator should be as simple as possible.

In the case where prediction is to be used in tracking a selected satellite or is associated with identification of the given satellite, additional requirements arise out of the need for solving related problems.

In practice, sets F_{0i} may be determined only in the statistical sense and are therefore, characterized by the distribution functions for the parameters $D_{0i} = D_i(t_0, x_1^0, \ldots, x_N^0)$. Application of the operator S_{ti} replaces D_{0i} with some other distribution function $D_{ti} = D_i(t, x_1, \ldots, x_N)$, which is defined as /2

$$D_{ti} = \varphi(S_{ti})D_{0i}. \quad (17.2)$$

Consequently, the result of prediction, even in the case of a regular operator S_{ti}, is statistical, and it is difficult to find the form of distribution function D_{ti} in view of the nonlinearity of the prediction operator.

The problem is complicated still further by the fact that the operator S_{ti} is not regular even in the case of strict formulation. Actually, if we recall for instance, that the parameters of the Earth's gravitational field are known with a random error (since they are determined statistically from a number of measurements) or that the parameters of the upper atmosphere may fluctuate in a random manner, then the coefficients in the system of differential equations (which comprise the operator S_{ti}) must also be statistical in nature.

At the present time, there is no known analytical method for solving the Cauchy problem for a system of nonlinear equations with random coefficients. Therefore, the following two methods may be mentioned for solving the prediction problem.

One of them consists of determining the parameters (which are random quantities) from observations of motion together with the initial conditions necessary for solving the Cauchy problem. Therefore, they are included in the statistical set F_{0i} and the operator S_{ti} itself becomes regular.

In this way we may account, for instance, for the random nature of variation in the density of the upper atmosphere and in the aerodynamic characteristics of the space vehicle under consideration. In this case, some generalized parameter, which unites both the above-mentioned quantities, may be statistically determined together with the motion parameters from information obtained on the basis of observations and may be included in the set F_{0i}, which now becomes more than just the set of initial conditions.

A priori data on random parameters in many instances may carry considerably more statistical information than a posteriori data determined during the limited time of observation of the space vehicle (in particular, this will /223 take place if an attempt is made to refine the parameters of the geoid from information obtained over several sessions of observing an artificial Earth satellite). When the a priori statistics are sufficiently great, we may use the second method for solving the prediction problem, in which we utilize only a priori information on the random parameters appearing in operator S_{ti}. In this method, mathematical expectations for the random quantities are introduced into the prediction operator, and then one of the methods is used for estimating the errors in determining parameters F_{ti} due to random deviations of the coefficients of operator S_{ti} from their mathematical expectation.

Although the first method for solving the problem is mathematically more refined and stricter, it complicates considerably the process of determining the elements of set F_{0i}. The second method may be used for fairly readily solving the prediction problem, but under condition that the errors in F_{ti} which arise in this case are within limits which are satisfactory for us. In this case, the operator S_{ti} may be simplified by disregarding some of the forces which act on the space vehicle and estimating the resultant error in the elements of F_{ti} together with the errors due to substituting regular parameters for the random parameters of the differential equations. Thus, we may arrive at a compromise solution in which the operator is comparatively simple in form, while prediction errors are no greater than the predetermined values. For instance, if we assume that the prediction operator describes Keplerian motion of an artificial Earth satellite, we get the simplest form of S_{ti}; however, in this case, errors are introduced into the position of the satellite as determined, resulting not only from disregarding the random values of the parameters, (e.g. the error in acceleration due to gravity on the surface of the spherical Earth) but also from disregarding a number of effects which disturb Keplerian motion of the satellite.

By using the estimates derived in the preceding chapters and the method for numerical solution of the Cauchy problem in the case of motion in the field of various gravitational models, we may use this particular method for constructing the prediction operator. Thus, selection of the appropriate model of the Earth's field is determined by the permissible value of the systematic error. The prediction problem may be more quickly solved by using an algorithm based on any of the analytical theories as the operator S_t Again, however, this is possible only when the systematic error is within the permissible value. /224

In solving problems involving identification of the satellite as well as prediction, it may be advisable to use an algorithm constructed in a system of osculating parameters. Since the orbits of artificial satellites designed for solving special scientific problems may have the most diverse eccentricities (including those nearly equal to zero), preference should be given in solving such problems to using a system of parameters such as Ω, i, p, q, k, t (with argument u) or Ω, i, p, q, k, u (with argument t) or similar systems of parameters (i.e. without singularities over the entire range of eccentricities).

The number of artificial satellites and various objects (such as the final stages of rockets) revolving in satellite orbits increases continuously with time. In this situation, the necessity for systematic observation of satellites designed for some scientific research or national economic assignment will involve the necessity for distinguishing them from other of less interest at the given time, i.e. for checking the parameters of satellites under observation (or their trajectories) against known values[1]. The large number of space objects may require nearly continuous comparison and reference to the catalog. Since the parameters of the observed satellite are determined in some single instant of time, while the parameters of other satellites (recorded in the catalog) with which the comparison is being made are referred to other moments of time, the process of identification will inevitably involve the necessity for nearly incessant repetition of motion prediction (determination of the orbital parameters). Therefore, it is advisable in selecting the operator S_t in a given instance to look toward simplification of cataloging and identifying the satellites as well as simplification of the solution of prediction problems. For instance, the use of osculating parameters may permit carrying out identification and prediction in several stages, which in turn may save time and simplify the general algorithm for solving the entire problem as a whole. /225

As an example, let us examine the manner in which such a complex problem might be solved using the results described above. In this case, the values of the orbital parameters of cataloged satellites could be arranged in increasing order in each of the corresponding seven groups:

$$\Omega_0,\ i_0,\ p_0,\ q_0,\ k_0,\ t_0,\ u_0. \tag{17.3}$$

In this case, the satellite under observation may be identified with one of those cataloged in several stages with successively increasing accuracy.

In the first stage, elements (17.3) are assumed to be constant (with the exception of t and u). In other words the prediction operator corresponds to Keplerian motion in the given case.

By successive comparison (with respect to the groups of parameters indicated in (17.3))between the orbital elements of the satellite under investigation and those recorded in the catalog and by eliminating satellites with "unsuitable" values of these quantities, a certain number of satellites may be isolated which are listed in the catalog and which have orbital parameters close to those of the given satellite. The algorithm for sorting parameters in the first stage of comparison may be most advantageously arranged to save time by taking account of experimental entropy.

Participating in the second stage of comparison are only those satellites which were retained after the first stage (it may be assumed that there will be comparatively few of them). The prediction operator S_{ti} in this case may be constructed on the basis of one of the algorithms given in the third chapter. As a result of the second comparison, the satellite under

[1] These parameters may be summarized in catalogs stored in the memory units of information machines.

investigation may be identified with one of those contained in the catalog (if the orbital parameters predicted by operator S_{ti} and those of the satellite in the catalog agree with the required accuracy), or it may be found that this satellite has not been previously cataloged. It is also possible that the predicted parameters of the orbits of several satellites will be close to the given parameters after completion of comparison in the second stage.

In the latter case, a third checking stage is utilized (the number of satellites to be compared in this case will obviously be fewer than in the second stage) in which an operator S_{ti} which accounts for all possible disturbances is used for prediction. In this case, utilization of the operator reduces to numerical solution (on a digital computer) of the Cauchy problem for the system of differential equations of the given type. /226

The given example of possible joint solution of the problems of prediction and identification shows how the form of the operator S_{ti} and the system of parameters in which the prediction problem is solved may be selected for simplifying the algorithm and reducing machine time.

§18. Effect of Inaccuracy in the Values of Geophysical Constants

The presently existing possibilities and methods for determining the constants which characterize the force field and figure of the Earth inevitably entail errors in the determination of these constants. It is of interest to evaluate the uncertainty contributed by these errors to the calculated satellite motion.

An analysis of this type has already been given by I. M. Yatsunskiy [81]. In this section we shall give relationships derived in another way which are fairly simple and convenient for qualitative analysis and approximate computations.

Basing our discussion on the estimate of the effect of higher terms in the potential expansion (which, as has already been pointed out, depend on such comparatively "fine" shades of difference as higher orders of polar oblateness, equatorial flattening, hemispheric asymmetry, irregular distribution of the Earth's mass, etc.), we shall limit ourselves in the given case to consideration of model Б. And, so we shall find out how errors in determining the equatorial radius of the Earth (r_0), polar flattening (α) and the constant $\mu = fM$ affect satellite motion in the field of a spheroid which is described by the zeroth and second zonal harmonics in the expansion of the potential [1].

---

[1] The angular velocity of the Earth's rotation is now known fairly accurately (to six significant decimal places), and therefore the error in this constant may be disregarded. /226

Let us use for this purpose the solutions derived in §11 (from the results of §16, we know the precision with which true motion is represented by these solutions). For the sake of simplicity and uniformity in obtaining an estimate, we shall assume that the second and third equations of (11.1) are written as usual with respect to the functions i and p. This has no appreciable adverse effect on the accuracy of our results. The corresponding solutions in this case derived in place of the second and third equations in (11.9)) will take the form /227

$$i = i_0+\frac{1}{2}\varepsilon\sin 2i_0\left[\frac{1}{2}\Delta(\sin^2 u)-\frac{1}{3}q_0\Delta(\cos^3 u)+\frac{1}{3}k_0\Delta(\sin^3 u)\right];$$

$$p = p_0+2p_0\varepsilon\sin^2 i_0\left[\frac{1}{2}\Delta(\sin^2 u)-\frac{1}{3}q_0\Delta(\cos^3 u)+\frac{1}{3}k_0\Delta(\sin^3 u)\right].$$

Now the equations with respect to all five functions $\Omega(u)$, $i(u)$, $p(u)$, $q(u)$ and $k(u)$ may be written in the identical form

$$F = F_0 + \varepsilon F^*(u). \tag{18.1}$$

Here, F_0 = const, while $F^*(u)$ is conditionally assumed to be dependent only on the angular argument u. All the geophysical constants named above appear only in the parameter $\varepsilon = 3c_{20}\dfrac{r_0^2}{p_0^2}$

On the basis of (5.9)

$$c_{20} = -\frac{2}{3}\left(\alpha-\frac{1}{2}m\right),$$

while m, on the basis of §5 (with an accuracy to quantities of the first order with respect to α), is equal to

$$m = \omega^2 r_0^3/\mu.$$

Thus

$$\varepsilon = -2\frac{r_0^2}{p_0^2}\left(\alpha - \frac{1}{2}m\right).$$

Partial differentiation with respect to α, r_0 and μ gives us

$$\left.\begin{aligned}\frac{\partial\varepsilon}{\partial\alpha} &= -2\frac{r_0^2}{p_0^2};\\ \frac{\partial\varepsilon}{\partial r_0} &= 2\frac{\varepsilon}{r_0} + 3m\frac{r_0}{p_0^2};\\ \frac{\partial\varepsilon}{\partial\mu} &= -\frac{m}{\mu}\frac{r_0^2}{p_0^2}.\end{aligned}\right\} \quad (18.2)$$

The logarithmic derivatives of expressions (18.1) are equal to /228

$$\begin{aligned}\frac{1}{F-F_0}\frac{\partial(F-F_0)}{\partial\alpha} &= \frac{\partial\varepsilon}{\partial\alpha}\varepsilon^{-1};\\ \frac{1}{F-F_0}\frac{\partial(F-F_0)}{\partial r_0} &= \frac{\partial\varepsilon}{\partial r_0}\varepsilon^{-1};\\ \frac{1}{F-F_0}\frac{\partial(F-F_0)}{\partial\mu} &= \frac{\partial\varepsilon}{\partial\mu}\varepsilon^{-1}\end{aligned}$$

and from the standpoint of linear theory (which is valid in view of the smallness of errors $\delta\alpha$, δr_0, $\delta\mu$) we may write

$$\left.\begin{aligned}\delta\bar{F}_\alpha &= \frac{\delta F_\alpha}{F-F_0} = \frac{\partial\varepsilon}{\partial\alpha}\varepsilon^{-1}\delta\alpha;\\ \delta\bar{F}_{r_0} &= \frac{\delta F_{r_0}}{F-F_0} = \frac{\partial\varepsilon}{\partial r_0}\varepsilon^{-1}\delta r_0;\\ \delta\bar{F}_\mu &= \frac{\delta F_\mu}{F-F_0} = \frac{\partial\varepsilon}{\partial\mu}\varepsilon^{-1}\delta\mu.\end{aligned}\right\} \quad (18.3)$$

The quantities $\delta\overline{F}_{\alpha}$, $\delta\overline{F}_{r_0}$ and $\delta\overline{F}_{\mu}$, as well as the post-orbital relative errors in the orbital elements, represent the error in the element due to uncertainty in the corresponding geophysical constant referred to the disturbance of this element by the second zonal harmonic of the potential.

Substituting (18.2) in (18.3) we get (with regard to the fact that $c_{20} < 0$)

$$\delta\overline{F}_{\alpha} = -2\frac{r_0^2}{p_0^2}\varepsilon^{-1}\delta\alpha = \frac{2}{3}\frac{\delta\alpha}{c_{20}};$$

$$\delta\overline{F}_{r_0} = \left(2\frac{\varepsilon}{r_0} + 3m\frac{r_0}{p_0^2}\right)\varepsilon^{-1}\delta r_0 = \left(2 - \frac{m}{c_{20}}\right)\frac{\delta r_0}{r_0};$$

$$\delta\overline{F}_{\mu} = -\frac{m}{\mu}\frac{r_0^2}{p_0^2}\varepsilon^{-1}\delta\mu = \frac{m}{3c_{20}}\frac{\delta\mu}{\mu}.$$

The expressions

$$\left.\begin{aligned} \delta\overline{F}_{\alpha} &= \frac{2}{3}\frac{\delta\alpha}{c_{20}}; \\ \delta\overline{F}_{r_0} &= \left(2 - \frac{m}{c_{20}}\right)\frac{\delta r_0}{r_0}; \\ \delta\overline{F}_{\mu} &= \frac{m}{3c_{20}}\frac{\delta\mu}{\mu} \end{aligned}\right\} \tag{18.4}$$

/229

solve the given problem (the constant c_{20} is taken here with a positive sign). As may be seen from these expressions, the relationships for errors in orbital elements (as a function of the angular argument u) behave similarly to the relationships for the perturbations of these elements. Consequently, it may be concluded on the basis of the results of §7 that errors in the values of the geophysical constants will have a maximum effect on the focal parameter and inclination for values of the argument $u = 90°$ and $u = 270°$ (assuming $u_0 = 0$); the effect on the longitude of the ascending node is greatest when $u = 360°$, etc.

Since $\alpha \simeq m$ (see §5; the difference between them is equal to approximately 3% of α), if we assume $\alpha = m$ we get

$$|c_{20}| = \frac{2}{3}\left(\alpha - \frac{1}{2}m\right) = \frac{1}{3}\alpha.$$

Therefore, from (18.4) we have

$$\left.\begin{aligned} \delta\bar{F}_{\alpha} &= 2\,\delta\alpha/\alpha; \\ \delta\bar{F}_{r_0} &= -\,\delta r_0/r_0; \\ \delta\bar{F}_{\mu} &= \delta\mu/\mu. \end{aligned}\right\} \tag{18.5}$$

with an accuracy completely within the tolerance for the given estimates. Relationships (18.5) show that the relative errors in the values of r_0 and μ have an identical effect (with respect to numerical value) on the relative errors in the orbital elements; the effect of the error in oblateness (α) is twice as great. It follows from the second relationship in (18.5) that δF_{r_0} and δr_0 vary with opposite signs. The reason for this becomes clear if it is remembered that a positive value of δr_0 is equivalent to an increase in radius r_0, and when the values of α and μ are the same, this should result in a reduction in $|c_{20}|$, which in the given case characterizes the entire disturbing effect of the aspherical Earth.

The maximum relative errors $\delta\alpha/\alpha$, $\delta r_0/r_0$ and $\delta\mu/\mu$ may presently be given the following approximate values

$$\left.\begin{aligned} \delta\alpha/\alpha &= 0.18\cdot 10^{-3}; \\ \delta r_0/r_0 &= 0.136\cdot 10^{-4}; \\ \delta\mu/\mu &= 0.301\cdot 10^{-4}. \end{aligned}\right\} \tag{18.6}$$

Given in Table 38 are the maximum values of the absolute errors in the elements $\delta\Omega$, δi, δp, δq and δk calculated over the interval of a single revolution for an orbit with parameters: $\Omega_0 = 0$, $i_0 = 45°$, $p_0 = 6{,}996{,}086.8$ m and $e_0 = 0.04990338$, $\omega_0 = 0$, $u_0 = 0$ (variant 7 from Table 10) assuming the errors in the constants given in (18.6). Also given in this table is the value of Δr calculated from formula (6.10). /230

In the given case, the values of $\delta\Omega_0$, δi_0, δp_0, δq_0 and δk_0 were all taken as maximum, which in reality cannot take place. Therefore, Δr is a five-dimensional sphere within which the parameters of the orbit will always be located for the given errors in the geophysical constants.

TABLE 38 *

Effect	$\delta\Omega$	δi	δp, м	$\delta q \cdot 10^6$	$\delta k \cdot 10^6$	Δr, m
$\delta\alpha/\alpha$	0″,50	0″,050	3,60	0,65	0,740	23,0
$\delta r_0/r_0$	0″,03	0″,004	0,27	0,05	0,056	1,3
$\delta\mu/\mu$	0″,07	0″,008	0,60	0,11	0,230	3,5

*Tr. Note: Commas indicate decimal points.

The figures in the last column show that errors in the geophysical constants have very little effect on the trajectory of the satellite over the interval of a single reovlution.

§19. Dispersion of Ballistic Trajectories

The uncertainty of a number of factors which affect rocket motion (aerodynamic characteristics of the rocket, atmospheric density, parameters of the gas in the combustion chamber, etc.), and the impossibility of absolute accuracy in maintaining controlled quantities (for instance errors in the position of the thrust vector and the engine cutoff time) lead to errors in the phase coordinates[1] at the end of the powered segment. This in turn results in dispersion of ballistic trajectories, i.e. the trajectories of rockets and satellites over flight segments unaffected by active forces. The value of this dispersion gives information necessary for evaluating the possibility of carrying out a planned study program with application to the given space vehicle.

In the mathematical sense, determination of the dispersion of ballistic trajectories is no different from the problem described by relationships (17.2) and (17.1), assuming a statistically given set F_{0i} and also (strictly /231
speaking) a statistical operator S_{ti}. Disregarding uncertainty as to the forces acting over the segment of ballistic motion, it may be assumed that the operator S_{ti} is regular.

The computation involved in this problem may be carried out with adequate rigor only on a digital computer, and is based on the theory of statistical solutions. The diagrammatic structure of the algorithm may be represented

[1] That is, in the coordinates and velocities of the rocket's center of gravity.

as follows:

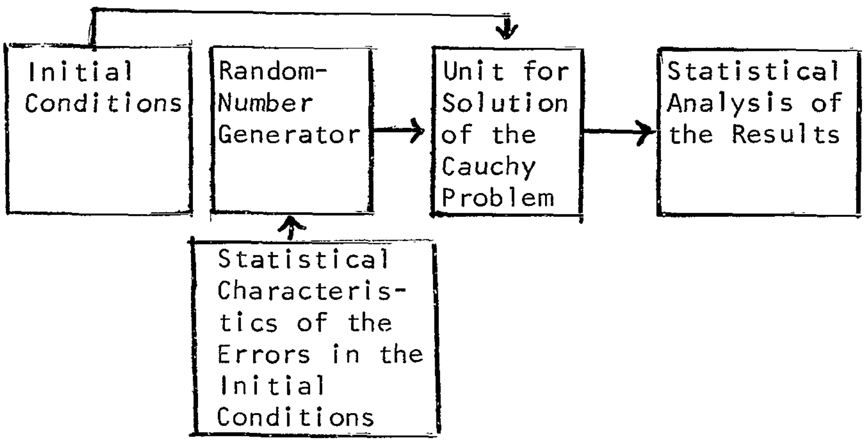


Fig. 116

The statistical law which governs distribution of the errors at the end of the powered segment (Gaussian distribution is assumed as a rule) and the numerical characteristics of this law (the matrix of the second moments) serve as the initial data for the random number unit. On the basis of this information, this unit generates the values of errors in the initial conditions which conform to the distribution law with predetermined characteristics. The errors are added to the nominal values of the initial conditions and fed to the unit for solution of the Cauchy problem. Thus, each solution is a random realization.

Another possibility is to feed the matrix for the second moments of the initial conditions (rather than the errors) to the random number unit.

By giving the reliability of the results in terms of a numerical characteristic (confidence limits, see for instance [82, 83]), we find the necessary number of random realizations (solutions of the Cauchy problem) which satisfy the initial conditions as given by the random number unit. Statistical analysis of these solutions[1] provides us with a basis for determining the mathematic expectation for the position of the terminal point in the space of /232
the given parameters, and for finding the matrix of the second moments.

This method of solution involves a rather large expenditure of machine time (e.g. more than 50 realizations are needed to obtain a result with a

---

1 It is arbitrarily assumed that the solutions also conform to Gaussian law, even though this law, strictly speaking, is deformed after solution of the Cauchy problem due to nonlinearity of the operator S_t.

reliability of approximately 0.9). In a number of calculations, particularly those of a preliminary nature, it is important to have a procedure for accurate but fairly simple determination of the boundaries of the maximum region in which the multidimensional tube of the trajectories will be located for a given dispersion of the initial conditions. It is known that in the case where the statistical variable conforms to Gaussian distribution law with standard deviation σ^2, the difference between this variable and the mathematical expectation is no greater than 3σ with a probability of 0.9973. A deviation equal to 3σ is ordinarily called maximum.

Assuming that the standard deviation (and consequently the maximum deviations) of the initial conditions is low, we may write within the limitations of linear theory

$$\delta F_t = \frac{\partial S_t}{\partial F}\,\delta F_0. \tag{19.1}$$

The symbol $\partial S_t/\partial F_0$ designates the operator obtained after differentiating S_t with respect to the elements of the set F; ∂F_t and ∂F_0 are the final and initial maximum deviations respectively for the elements of set F.

Thus, we may use any of the analytical theories described in the third chapter for finding the dispersion of trajectories over the ballistic segment with regard to the effect of eccentricity of the Earth's gravitational field. In this connection, it is advisable to select a simpler algorithm (a perfectly natural desire), and to restrict ourselves within this algorithm to the most appreciable effect of eccentricity, i.e. the effect of the second zonal harmonic. Actually, the contributions to the result are vanishingly small when the part of operator S_t which accounts for the effect of the higher harmonics of the potential expansion (these parts enter S_t additively; see Chapter Three) is applied to the deviations in the initial conditions.

Assuming for instance, that the prediction operator is constructed according to the algorithm described in §11, we may expand expression (19.1) into the following set of equations:

$$\delta\Omega = \frac{\partial\Omega}{\partial\Omega_0}\delta\Omega_0+\frac{\partial\Omega}{\partial i_0}\delta i_0+\frac{\partial\Omega}{\partial p_0}\delta p_0+\frac{\partial\Omega}{\partial q_0}\delta q_0+\frac{\partial\Omega}{\partial k_0}\delta k_0+\frac{\partial\Omega}{\partial u_0}\delta u_0; \tag{19.2}$$

$$\delta i = \frac{\partial i}{\partial\Omega_0}\delta\Omega_0+\frac{\partial i}{\partial i_0}\delta i_0+\frac{\partial i}{\partial p_0}\delta p_0+\frac{\partial i}{\partial q_0}\delta q_0+\frac{\partial i}{\partial k_0}\delta k_0+\frac{\partial i}{\partial u_0}\delta u_0;$$ /233

$$\left.\begin{aligned}
\delta p &= \frac{\partial p}{\partial \Omega_0}\delta\Omega_0+\frac{\partial p}{\partial i_0}\delta i_0+\frac{\partial p}{\partial p_0}\delta p_0+\frac{\partial p}{\partial q_0}\delta q_0+\frac{\partial p}{\partial k_0}\delta k_0+\frac{\partial p}{\partial u_0}\delta u_0;\\
\delta q &= \frac{\partial q}{\partial \Omega_0}\delta\Omega_0+\frac{\partial q}{\partial i_0}\delta i_0+\frac{\partial q}{\partial p_0}\delta p_0+\frac{\partial q}{\partial q_0}\delta q_0+\frac{\partial q}{\partial k_0}\delta k_0+\frac{\partial q}{\partial u_0}\delta u_0;\\
\delta k &= \frac{\partial k}{\partial \Omega_0}\delta\Omega_0+\frac{\partial k}{\partial i_0}\delta i_0+\frac{\partial k}{\partial p_0}\delta p_0+\frac{\partial k}{\partial q_0}\delta q_0+\frac{\partial k}{\partial k_0}\delta k_0+\frac{\partial k}{\partial u_0}\delta u_0.
\end{aligned}\right\} \tag{19.2}$$

All partial derivatives in (19.2) are obtained by differentiating equations (11.9) with respect to the initial values of the parameters; the quantities $\delta\Omega_0$, δi_0, δp_0,... are the errors in the initial conditions (the set δF_0). Since the partial derivatives are functions of the angular quantity u, the functions $\delta\Omega$, δi, δp,... describe dispersion of the ballistic trajectories (the set δF_t).

If the set δF_0 is given in the form of maximum errors, then the tube δF_t is also a maximum. The awkwardness of the expressions for $\partial\Omega/\partial\Omega_0$, $\partial\Omega/\partial i_0$,..., $\partial i/\partial\Omega_0$, $\partial i/\partial i_0$,..., etc. is offset by the fact that equations (19.2) may be used without recourse to solving the Cauchy problem for determining dispersion at any point of the tube with regard to eccentricity of the Earth's gravitational field.

Equations (19.2) describe the dispersion of orbits; in order to find the dispersion in the position of the satellite itself, equations (19.2) must be supplemented by an expression of the form

$$\delta t=\delta t(\delta\Omega_0, \delta i_0, \delta p_0, \delta q_0, \delta k_0, \delta u_0, \Omega_0, i_0, p_0, q_0, k_0, u_0; u) \tag{19.3}$$

or

$$\delta u=\delta u(\delta\Omega_0, \delta i_0, \delta p_0, \delta q_0, \delta k_0, \delta u_0, \Omega_0, i_0, p_0, q_0, k_0, t_0; t). \tag{19.4}$$

In case (19.3) the argument is an angle (e.g. the argument of latitude u), while in the second case the argument is time t.

Relationship (19.3) may be represented in the form of the equation

$$\delta t = t^i\delta i_0 + t^p\delta p_0 + t^q\delta q_0 + t^k\delta k_0 + t^u\delta u_0, \tag{19.5}$$

where

$$\left.\begin{aligned} t^i &= \frac{\partial t}{\partial i} i^i + \frac{\partial t}{\partial p} p^i + \frac{\partial t}{\partial q} q^i + \frac{\partial t}{\partial k} k^i; \\ t^p &= \frac{\partial t}{\partial i} i^p + \frac{\partial t}{\partial p} p^p + \frac{\partial t}{\partial q} q^p + \frac{\partial t}{\partial k} k^p; \\ & \cdots\cdots\cdots\cdots\cdots \\ & \cdots\cdots\cdots\cdots\cdots \\ t^u &= \frac{\partial t}{\partial u_0}. \end{aligned}\right\} \tag{19.6}$$

/234

The partial derivatives in (19.6) are obtained by differentiating the known expression (11.10):

$$\left.\begin{aligned} \frac{\partial t}{\partial i} &= -\frac{1}{\sqrt{\mu}} \int_{u_0}^{u} \frac{\sin 2i \sin^2 u}{\sqrt{p}\, R} du; \\ \frac{\partial t}{\partial p} &= \frac{1}{\sqrt{\mu}} \left[\frac{3}{2} \int_{u_0}^{u} \frac{p^{1/2}}{R^2} du - \frac{1}{2} \varepsilon p_0^2 \int_{u_0}^{u} \frac{\cos^2 i \sin^2 u}{p^{3/2} R} du \right]; \\ \frac{\partial t}{\partial q} &= -\frac{1}{\sqrt{\mu}} \left[2 \int_{u_0}^{u} \frac{p^{3/2} \cos u}{R^3} du - \varepsilon p_0^2 \int_{u_0}^{u} \frac{\cos^2 i \sin^2 u \cos u}{\sqrt{p}\, R^2} du \right]; \\ \frac{\partial t}{\partial k} &= -\frac{1}{\sqrt{\mu}} \left[2 \int_{u_0}^{u} \frac{p^{3/2} \sin u}{R^3} du - \varepsilon p_0^2 \int_{u_0}^{u} \frac{\cos^2 i \sin^3 u}{\sqrt{p}\, R^2} du \right]; \\ \frac{\partial t}{\partial u_0} &= -\frac{1}{\sqrt{\mu}} \left[\frac{p_0^{3/2}}{(1+q_0 \cos u_0 + k_0 \sin u_0)^2} - \varepsilon p_0^{3/2} \frac{\cos^2 i_0 \sin^2 u_0}{1+q_0 \cos u_0 + k_0 \sin u_0} \right]. \end{aligned}\right\} \tag{19.7}$$

Here ε is a small parameter (see §11), $R = 1 + q \cos u + k \sin u$, and the functions i, p, q and k in the integrands are to be substituted from (11.9). The symbols i^i, p^i,..., i^p, p^p,..., etc. denote partial derivatives $\partial i/\partial i_0$, $\partial p/\partial i_0$,..., $\partial i/\partial p_0$, $\partial p/\partial p_0$,..., etc. which also appear in relationships (19.2). In computing expressions (19.6), they should be placed under the integral signs in (19.7), although in (19.6) they are arbitrarily represented as the products

$$\frac{\partial t}{\partial i} i^i, \frac{\partial t}{\partial p} p^i, \frac{\partial t}{\partial q} q^i, \ldots$$

Thus, the computation of dispersion δt from formula (19.5) reduces to numerical quadrature of relationships (19.7), the integrands of these relationships being multiplied by the partial derivatives $\partial i/\partial i_0$, $\partial i/\partial p_0, \ldots$, while /235 functions i, p, q and k are given by equations (11.9). This method is undoubtedly awkward, and it may be preferable to compute δt as the difference between integrals (11.10) taken for the nominal values of the initial parameters and the same integrals taken for the "divergent" values (with regard to errors δi_0, $\delta p_0, \ldots$).

I. EXPANSION OF THE GRAVITATIONAL POTENTIAL OF THE EARTH IN A SERIES WITH RESPECT TO SPHERICAL FUNCTIONS

An infinite series representing the gravitational potential of the Earth may be derived most naturally by finding the expansion with respect to certain functions directly related to every particular problem to be considered and to what might be called the specific essence of the process being studied. These functions are called the eigenfunctions of the problem.

First, let us assure ourselves that potential (2.4) is a harmonic function, i.e. that it satisfies the Laplace equation

$$\Delta V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0. \tag{I.1}$$

It is sufficient for this purpose to differentiate (2.4) with respect to x, y and z as parameters, and to substitute the resultant expressions

$$\frac{\partial^2 V}{\partial x^2} = -f \int\limits_{x_m} \int\limits_{y_m} \int\limits_{z_m} \rho \left[\frac{1}{r^3} - \frac{3(x-x_m)^2}{r^5}\right] dx_m dy_m dz_m;$$

$$\frac{\partial^2 V}{\partial y^2} = -f \int\limits_{x_m} \int\limits_{y_m} \int\limits_{z_m} \rho \left[\frac{1}{r^3} - \frac{3(y-y_m)^2}{r^5}\right] dx_m dy_m dz_m;$$

$$\frac{\partial^2 V}{\partial z^2} = -f \int\limits_{x_m} \int\limits_{y_m} \int\limits_{z_m} \rho \left[\frac{1}{r^3} - \frac{3(z-z_m)^2}{r^5}\right] dx_m dy_m dz_m$$

in (I.1)[1]

[1] In the case where (x,y,z) is the internal point of a region, the function V satisfies the Poisson equation $\Delta V = -4\pi f\rho$. In this case the function $\rho(x_m, y_m, z_m)$ is required to be piecewise-continuously differentiable in the given region.

In this way, we are convinced that (2.4) is the solution of the Laplace equation[1]. /23

Let us now present this solution as a series in eigenfunctions. For this purpose we take the relationships

$$\left.\begin{aligned} x &= r\sin\varphi\cos\lambda; \\ y &= r\sin\varphi\sin\lambda; \\ z &= r\cos\varphi, \end{aligned}\right\} \qquad (I.1')$$

and convert to spherical coordinates: radius r, spherical latitude ϕ and longitude λ, in which the Laplace equation is written as

$$\Delta V(r,\varphi,\lambda) = \frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{\sin\varphi}\frac{\partial}{\partial\varphi}\left(\sin\varphi\frac{\partial V}{\partial\varphi}\right) + \frac{1}{\sin^2\varphi}\frac{\partial^2 V}{\partial\lambda^2} = 0. \qquad (I.2)$$

The problem in which we are interested (the so-called first external boundary problem or the external Dirichlet problem for equation (I.2)) may be formulated as the problem of finding the function which satisfies equation (I.2) in the space outside some closed surface (while simultaneously satisfying the previously mentioned condition at infinity) and takes on given values at the boundary. We shall take sphere Σ with radius r_0 as the boundary surface.

In determining the form of the terrestrial potential, the function V is sought outside the volume bounded by the surface of the Earth, which is not spherical, but is nearly spherical. Therefore, r_0 may be understood to mean the greatest radius of the Earth (roughly--the semimajor axis of the equatorial ellipse). The boundary conditions are given on this sphere, and a solution is sought which satisfies the boundary conditions given on the surface of the Earth.

The validity of this approach is explained by the stability of the solutions the boundary of the region varies (this point is taken up in detail in [84]; the intuitive premises for the given approach are not only in the similarity between the selected sphere and the Earth's surface, but also in the fact that the constants appearing in the solution are actually determined by measurements made on the real surface of the Earth.

The condition for function V on the surface of Σ is /2

$$V(r,\varphi,\lambda)\big|_{\Sigma} = \mu/r_0.$$

[1] It is also required for uniqueness of the solution for the external Dirichlet problem that the function V must satisfy the condition at infinity $\underset{r\to\infty}{V\to 0}$. It is immediately clear from (2.2) that the potential of a uniform sphere satisfies this requirement. The same may be shown for potential (2.4) (see [31, 44, 45, 1, 2]. /23

The remaining boundary conditions reduce to requirements for the continuity and differentiability of the function V on the sphere.

Actually, since the surface Σ is closed, we cannot speak of the values of V at some ϕ and λ singled out as boundary values; we should only apply the conditions of continuity and differentiability of the solution at the ends of the interval

$$0 \leq \lambda < 2\pi, \quad 0 \leq \varphi \leq \pi,$$

i.e. we should require fulfillment of the conditions

$$V(\lambda = 0) = V(\lambda = 2\pi); \quad \left(\frac{\partial V}{\partial \lambda}\right)_{\lambda = 0} = \left(\frac{\partial V}{\partial \lambda}\right)_{\lambda = 2\pi} \tag{I.3}$$

and [1]

$$\left.\begin{array}{l} |\lim\limits_{\varphi \to 0} V| < \infty; \\ |\lim\limits_{\varphi \to \pi} V| < \infty. \end{array}\right\} \tag{I.4}$$

Relationship (I.3) sets the condition of periodicity for function V, while (I.4) sets the condition of boundedness.

In seeking the solution for equation (I.2), we assume that

$$V(r, \varphi, \lambda) = R(r) Y(\varphi, \lambda). \tag{I.5}$$

After substituting (I.5) in (I.2), the variables separate and (I.2) may be written as two equal relationships, one of which depends only on r, and the other only on ϕ and λ. Therefore, they may be set equal to some constant

[1]The Laplace operator is invariant to orthogonal transformations of the coordinate system. Therefore, the singularity of equation (I.2) at the points $\phi = 0$, $\phi = \pi$ is due only to selection of the given coordinate system. Inequality (I.4) requires the function V to behave at the poles as it does on the rest of the sphere.

$$\frac{2rR'+r^2R''}{R} = -\frac{\frac{1}{\sin\varphi}\frac{\partial}{\partial\varphi}\left(\sin\varphi\frac{\partial Y}{\partial\varphi}\right)+\frac{1}{\sin^2\varphi}\frac{\partial^2 Y}{\partial\lambda^2}}{Y} = \varkappa.$$

Hence we get for determining R(r) (the equation of Euler) /239

$$r^2R''+2rR'-\varkappa R=0 \tag{I.6}$$

and for determining $Y(\phi,\lambda)$ the equation in partial derivatives of elliptical type

$$\frac{1}{\sin\varphi}\frac{\partial}{\partial\varphi}\left(\sin\varphi\frac{\partial Y}{\partial\varphi}\right)+\frac{1}{\sin^2\varphi}\frac{\partial^2 Y}{\partial\lambda^2}+\varkappa Y=0. \tag{I.7}$$

The general solution of the first equation is written in the form

$$R(r)=C_1r^{\alpha_1}+C_2r^{\alpha_2}, \tag{I.8}$$

where C_1 and C_2 are arbitrary constants, while α_1 and α_2 will subsequently be determined as the roots of the quadratic equation

$$\alpha(\alpha+1)-\varkappa=0. \tag{I.9}$$

We shall now seek the values of parameter κ for which equation (I.7) has solutions which are not identically equal to zero and which satisfy boundary conditions (I.3) and (I.4). These solutions are called eigenfunctions or fundamental functions, while the corresponding values of κ are called the eignevalues of the given differential equation.

Let us again use the method of separating variables, representing $Y(\phi,\lambda)$ as

$$Y(\varphi,\lambda)=\Phi(\varphi)\Lambda(\lambda). \tag{I.10}$$

We may then write

$$-\frac{\Lambda''_{\lambda\lambda}}{\Lambda}=\frac{\frac{1}{\sin\varphi}\frac{\partial}{\partial\varphi}(\sin\varphi\,\Phi'_{\varphi})+\varkappa\Phi}{\sin^2\varphi\,\Phi}=\chi,$$

where χ is a parameter

As a result of separation, we get the two equations

$$\Lambda''_{\lambda\lambda}+\chi\Lambda=0 \tag{I.11}$$

and

$$\frac{1}{\sin\varphi}\frac{d}{d\varphi}(\sin\varphi\,\Phi'_{\varphi})+\left(\varkappa-\frac{\chi}{\sin^2\varphi}\right)\Phi=0. \tag{I.12}$$

The general periodic solution (sätisfying conditions (I.3)) of equation (I.11), written in the form

$$\Lambda(\lambda)=c_m\cos m\lambda+d_m\sin m\lambda, \tag{I.13}$$

may be found only when $\chi = m^2$. The subscript m associated with the constants c_m and d_m shows that they depend on the value of this parameter. /240

Since χ may take on any values $\chi \geq 0$, let us assume $\chi = 0$ and go in equation (I.2) from $\sin\phi$ to $\cos\phi$, and then to the new argument $x = \cos\phi$.

We then get (without changing the former notation for the unknown function Φ) the Legendre equation

$$\frac{d}{dx}\left[(1-x^2)\frac{d\Phi}{dx}\right]+\varkappa\Phi=0. \tag{I.14}$$

With a transition to the argument x, the region of variation in ϕ $0 \leq \phi \leq \pi$ corresponds to the interval $-1 \leq x \leq 1$, and boundary conditions (I.4) are equivalent to setting the unknown function equal to zero at the ends of this range.

Since the solution of equation (I.14) at $\kappa \neq 0$ is not expressed in elementary functions, we shall seek Φ in the form of a polynomial of degree n:

$$\Phi_n(x) = x^n + a_1 x^{n-1} + a_2 x^{n-2} + \ldots + a_n .$$

Substituting $\Phi_n(x)$ in (I.14), we get

$$\frac{d}{dx}\left[-nx^{n+1} - a_1(n-1)x^n + \ldots\right] + \varkappa\left[x^n + a_1 x^{n-1} + \ldots\right] = 0$$

or

$$-n(n+1)x^n - a_1 n(n-1)x^{n-1} + \ldots + \varkappa x^n + \varkappa a_1 x^{n-1} + \ldots = 0. \tag{I.15}$$

If $\Phi_n(x)$ is a solution, then equation (I.15) should be an identity. This is possible only in the case where the coefficients associated with each power of x in (I.15) are equal to zero.

Specifically, in this case $-n(n-1) + \kappa = 0$. Hence, it follows that

$$\varkappa = n(n+1). \tag{I.16}$$

For these values of κ (here $n = 0, 1, 2,..$) we get an infinite sequence of polynomials $\Phi_n(x)$ which are nontrivial solutions of equation (I.14). It is shown in the theory of differential equations [85-87] that Legendre polynomials $P_n(x)$ are such polynomials of degree n which satisfy the boundary conditions given above. They may be determined from the relationship (Rodrigues' formula)[1]

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n . \tag{I.17}$$ /241

Thus, the values of κ given in (I.16) are the eigenvalues of the problem under consideration for the Laplace equation (I.2) with given boundary conditions, and the Legendre polynomials are the eigenfunctions (for the special case $\chi = 0$).

[1] Expressions for Legendre polynomials are given in Appendix II. /240

These polynomials are known to have a number of unusual properties [31, 44, 45, 1, 2] to which we shall refer in the following discussion where necessary.

Assuming now that $\chi \neq 0$ in equation (I.12),we find that in this case (for eigenvalues (I.16)) the eigenfunctions of the problem are

$$P_{nm}(x) = (1-x^2)^{\frac{m}{2}} \frac{d^m}{dx^m} P_n(x). \tag{I.17'}$$

The validity of this is confirmed by direct substitution of (I.17) in (I.12). Polynomials $P_{nm}(x)$ are called associated Legendre functions.

It also follows from (I.17') that $m \leq n$ (P_n is a polynomial of degree n).

Let us make the inverse transition to the variable ϕ, and we shall represent the Legendre polynomials from now on in the form $P_{nm}(\cos\phi)$.

Returning to relationship (I.10), we note that for any fixed eigenvalue $\varkappa$ (i.e. for any fixed value of the number n), we may write $Y_{nm} = (c_{nm} \cos m\lambda + d_{nm} \sin m\lambda)P_{nm}(\cos\phi)$. The parameter m in this case may assume the value of any integer in the interval $0 \leq m \leq n$.

The functions $\cos m\lambda P_{nm}(\cos\phi)$ and $\sin m\lambda P_{nm}(\cos\phi)$ (for various m) are called basic or fundamental spherical functions of order n.

Expression (I.17) shows the particular part played by the basic Legendre polynomials with respect to the associated functions.

The number of fundamental functions (for given n and all values of m) is equal to 2n + 1, and they form a complete orthonormal system [86, 87]. Therefore, any eignefunction $Y_n(\phi,\lambda)$ of equation (I.7) may be represented in the form of a linear combination of all functions $Y_{nm}(\phi,\lambda)$ which go to make up this system.

That is, /242

$$Y_{nm}(\varphi,\lambda) = \sum_{m=0}^{n} Y_{nm}(\varphi,\lambda) = \sum_{m=0}^{n} (c_{nm}\cos m\lambda + d_{nm}\sin m\lambda)P_{nm}(\cos\varphi).$$

The function $Y_n(\phi,\lambda)$ in turn forms a complete orthonormal system. This gives us the opportunity to apply the expansion theorem of [86], according to which any continuous function which satisfies the boundary conditions of a given problem and has piecewise-continuous first and second derivatives may be expanded in a series in eigenfunctions.

Before writing this expansion, let us return to the solution of equation (I.6). According to (I.9), when $\kappa = n(n-1)$ we get two roots: $\alpha_1 = n$ and $\alpha_2 = -(n+1)$, which correspond to two special solutions: $C_1 = r^n$ and $C_2 = r^{-(n+1)}$. The first of these is not bounded as $r \to \infty$, and therefore cannot be used (it corresponds to the internal Dirichlet problem). If the constant C_2 is determined in the second solution according to the condition on the sphere

$$V(r, \varphi, \lambda)\big|_\Sigma = \mu/r_0,$$

then for function R(r) we get the expression

$$R(r) = \frac{\mu}{r}\left(\frac{r_0}{r}\right)^n.$$

Thus, the final series expansion of the potential $V(r,\phi,\lambda)$ is written as

$$V(r, \varphi, \lambda) = \frac{\mu}{r}\sum_{n=0}^{\infty}\sum_{m=0}^{n}\left(\frac{r_0}{r}\right)^n (c_{nm}\cos m\lambda + d_{nm}\sin m\lambda)\, P_{nm}(\cos\varphi).$$

This series may also be obtained by other methods. The most widely used is expansion of the integrand 1/r in powers of a small quantity. The ratio of the size of the gravitating body to the distance between this body and the point being attracted is such a small quantity. The coefficients of the expansion obtained in this way are Legendre polynomials. This method is used later (in Appendix III) in solving another problem.

II. Spherical Functions /243

The principal spherical functions (Legendre's polynomials) are computed by Rodrigues' formula

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2 - 1)^n.$$

The associated spherical functions are computed by the formula

$$P_{nm}(x)=(1-x^2)^{\frac{m}{2}}\frac{d^m}{dx^m}P_n(x)=\frac{(1-x^2)^{\frac{m}{2}}}{2^n n!}\quad\frac{d^{n+m}(x^2-1)^n}{dx^{n+m}}.$$

Some recurrence formulas which relate the spherical functions:

$$(n+1)P_{n+1}(x)-(2n+1)xP_n(x)+nP_{n-1}(x)=0;$$

$$\frac{d}{dx}P_{n+1}(x)=(2n+1)P_n(x)+\frac{d}{dx}P_{n-1}(x);$$

$$(2n+1)xP_{nm}(x)-(n-m+1)P_{n+1,m}(x)-(n+m)P_{n-1,m}(x)=0.$$

The spherical functions from $P_{00}(x)$ to $P_{90}(x)$ have the following form:

$P_{00}=1;$ $\quad P_{31}=\frac{3}{2}(5x^2-1)\sqrt{1-x^2};$

$P_{10}=x;$ $\quad P_{32}=15x(1-x^2);$

$P_{11}=\sqrt{1-x^2};$ $\quad P_{33}=15(1-x^2)^{3/2};$

$P_{20}=\frac{1}{2}(3x^2-1);$ $\quad P_{40}=\frac{1}{8}(35x^4-30x^2+3);$

$P_{21}=3x\sqrt{1-x^2};$ $\quad P_{41}=\frac{5}{2}(7x^2-3x)\sqrt{1-x^2};$

$P_{22}=3(1-x^2);$ $\quad P_{42}=\frac{15}{2}(7x^2-1)(1-x)^2;$

$P_{30}=\frac{1}{2}(5x^3-3x);$ $\quad P_{43}=105x(1-x^2)^{3/2};$

$P_{44}=105(1-x^2)^2;$ $\quad P_{60}=\frac{1}{16}(231x^6-315x^4+105x^2-5);$

$P_{50}=\frac{1}{8}(63x^5-70x^2+15x);$ $\quad P_{61}=\frac{21}{8}(33x^5-30x^3+5x)\sqrt{1-x^2};$ /244

$P_{51}=\frac{15}{8}(21x^4-14x^2+1)\sqrt{1-x^2};$ $\quad P_{62}=\frac{105}{8}(33x^4-18x^2+1)(1-x^2);$

$P_{52}=\frac{105}{2}(3x^3-x)(1-x^2);$ $\quad P_{63}=\frac{315}{2}(11x^3-3x)(1-x^2)^{3/2};$

$P_{53} = \frac{105}{2}(9x^2-1)(1-x^2)^{3/2}$; $\quad P_{64} = \frac{945}{2}(11x^2-1)(1-x^2)^2$;

$P_{54} = 945x(1-x^2)^2$; $\quad P_{65} = 10395x(1-x^2)^{5/2}$;

$P_{55} = 945(1-x^2)^{5/2}$; $\quad P_{66} = 10395(1-x^2)^3$;

$$P_{70} = \frac{1}{16}(429x^7 - 693x^5 + 315x^3 - 35x);$$

$$P_{71} = \frac{7}{16}(429x^6 - 495x^4 + 135x^2 - 5)\sqrt{1-x^2};$$

$$P_{72} = \frac{63}{8}(143x^5 - 110x^3 + 15x)(1-x^2);$$

$$P_{73} = \frac{315}{8}(143x^4 - 66x^2 + 3x)(1-x^2)^{3/2};$$

$$P_{74} = \frac{3465}{2}(13x^3 - 3x)(1-x^2)^2;$$

$$P_{75} = \frac{10395}{2}(13x^2 - 1)(1-x^2)^{5/2};$$

$$P_{76} = 135135x(1-x^2)^3;$$

$$P_{77} = 135135(1-x^2)^{7/2};$$

$$P_{80} = \frac{1}{128}(6435x^8 - 12012x^6 + 6930x^4 - 1260x^2 + 35);$$

$$P_{81} = \frac{9}{16}(715x^7 - 1001x^5 + 385x^3 - 35x)\sqrt{1-x^2};$$

$$P_{82} = \frac{315}{16}(143x^6 - 143x^4 + 33x^2 - 1)(1-x^2);$$

$$P_{83} = \frac{3465}{8}(39x^5 - 26x^3 + 3x)(1-x^2)^{3/2};$$

$P_{84} = \frac{10395}{8}(65x^4 - 26x^2 + 1)(1-x^2)^2$; /245

$$P_{85} = \frac{135135}{2}(5x^3 - x)(1-x^2)^{5/2};$$

$$P_{86} = \frac{135135}{2}(15x^2 - 1)(1-x^2)^3;$$

$$P_{87} = 2027025x(1-x^2)^{7/2};$$

$$P_{88} = 2027025(1 - x^2)^4;$$

$$P_{90} = \frac{1}{128}(12155x^9 - 25740x^7 + 18018x^5 - 4620x^3 + 315x);$$

$$P_{100} = \frac{1}{256}(46189x^{10} - 109395x^8 + 90090x^6 -$$

$$- 30030x^4 + 3465x^2 - 63).$$

Extensive tables of the associated Legendre's functions are given in [88].

If the potential function is written in spherical coordinates, then the argument $x = \cos\vartheta$, where ϑ is the angle between the radius vector of the point and the axis of rotation of the Earth, or $x = \sin\phi$, where ϕ is the angle between the radius vector of the point and the plane of the equator.

If the potential function is written in the geocentric inertial rectangular coordinate system (the z-axis directed along the axis of rotation of the Earth toward the north, the x- and y-axes lying in the plane of the equator, x being directed toward the point of the vernal equinox, and y completing the right-handed coordinate system), then

$$x = \frac{z}{r}, \text{ where } r = \sqrt{x^2 + y^2 + z^2}.$$

The functions sin $m\lambda$ and cos $m\lambda$ which figure in the potential expansion are found in this case from the equations

$$x = r\cos\varphi\cos(\lambda + s);$$
$$y = r\cos\varphi\sin(\lambda + s);$$
$$z = r\sin\varphi.$$

Here ϕ and λ are respectively the latitude (reckoned from the plane of the equator) and the longitude (reckoned from Greenwich), while s is local sidereal time.

III. EXPRESSING THE COEFFICIENTS IN THE EXPANSION OF THE EARTH'S GRAVITATIONAL POTENTIAL IN TERMS OF MOMENTS OF INERTIA

/246

Shown in Appendix I was conversion of integral (2.4) to infinite series (2.6). In order to express the first coefficients of this expansion in terms of moments of inertia, we return again to expression (2.4), writing it in the form

$$V = f \int_M \frac{dm}{\Delta}, \tag{2.4'}$$

here (see Fig. 117) Δ is the distance between the instantaneous point with element of mass dm and an external material point of unit mass

$$\Delta = \sqrt{r^2 + r_1^2 - 2rr_1 \cos\gamma};$$

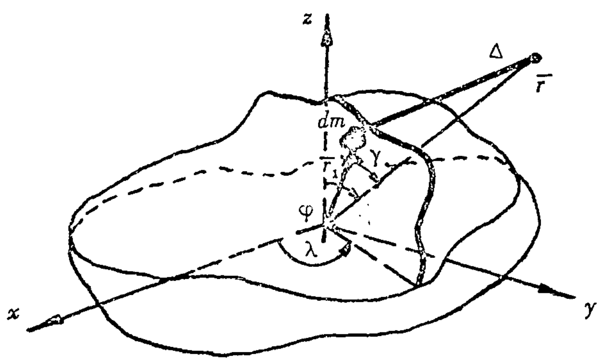


Fig. 117

r is the distance of the external point from the coordinate origin; r_1 is the instantaneous distance of the element of mass dm from the coordinate origin; γ is the angle $\widehat{rr_1}$ which may be determined from the scalar product

$$\cos\gamma = \frac{\vec{r}\cdot\vec{r}_1}{|\vec{r}\cdot\vec{r}_1|} = \frac{xx_1+yy_1+zz_1}{rr_1}$$ (III.1)

or in spherical coordinates (r,ϕ,λ)

$$\cos\gamma = \cos\varphi\cos\varphi_1 + \sin\varphi\sin\varphi_1\cos(\lambda-\lambda_1).$$ (III.1')

Since $r_1 < r$, the function

$$\frac{1}{\Delta} = \frac{1}{r\sqrt{1+\delta^2-2\delta\cos\gamma}} = \frac{1}{r}F(\delta)$$ (III.2)

may be expanded in a Taylor series with respect to powers of the small quantity $\delta = r_1/r$ (the steady convergence of this series is proved, for instance, in [31, 44, 45]. /247

As a result, we get

$$\frac{1}{\Delta} = \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{r_1}{r}\right)^n P_n(\cos\gamma).$$ (III.2')

The coefficients in this expression are Legendre polynomials[1] [31, 44, 45].

Substituting (III.2') in (2.4'), we get

$$V(r,\varphi,\lambda) = f\sum_{n=0}^{\infty}\frac{1}{r^{n+1}}\int_M r_1^n P_n(\cos\gamma)dm = \frac{fM}{r}\sum_{n=0}^{\infty}\left(\frac{r_0}{r}\right)^n Y_n.$$ (III.3)

The function V, as always, depends on the coordinates (in the given case, having in mind (III.1'), the spherical coordinates r, ϕ, λ) of the point on which it acts.

[1] The function $F(\delta)$ is therefore called a generating function of the Legendre polynomials.

It has already been pointed out in Appendix I that the spherical functions remain spherical after rotation of the coordinate axes (due to invariance of the Laplace equation with respect to orthogonal transformations).

By utilizing this fact, we may express any principal function in terms of the sum of the full set (for given n) of associated functions.

Actually, let the spherical coordinates of two points which are the ends of two vectors $\vec{r}$ and $\vec{r}_1$ be equal respectively, to (r,ϕ,λ) and (r_1, ϕ_1, λ_1).

If the reference axis for angles ϕ (axis oz) is turned through the angle γ in such a way as to bring it into coincidence with the vector $\vec{r}$, then the basic spherical function $P_n(\cos\gamma)$ may be represented in the form

$$P_n(\cos\gamma)=\sum_{m=0}^{n}\frac{2}{\delta_m}\frac{(n-m)!}{(n+m)!}P_{nm}(\cos\varphi)P_{nm}(\cos\varphi_1)\cos(\lambda-\lambda_1), \tag{III.4}$$

where $\delta_m = 2$ when $m = 0$, and $\delta_m = 1$ when $m \neq 0$. This expression is called a summation formula.

With regard to the summation formula, integral (III.3) is changed to

$$V(r,\varphi,\lambda)=\frac{fM}{r}\sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{r_0^n}{r^n}(c_{nm}\cos m\lambda+d_{nm}\sin m\lambda)P_{nm}(\cos\varphi), \tag{III.5}$$ /248

where

$$\left.\begin{aligned} c_{nm} &= \frac{1}{r_0^n M}\frac{2(n-m)!}{\delta_m(n+m)!}\int\limits_M r_1^n P_{nm}(\cos\varphi_1)\cos m\lambda_1\,dm;\\ d_{nm} &= \frac{1}{r_0^n M}\frac{2(n-m)!}{\delta_m(n+m)!}\int\limits_M r_1^n P_{nm}(\cos\varphi_1)\sin m\lambda_1\,dm.\end{aligned}\right\} \tag{III.6}$$

The integrals

$$c'_{nm}=\int\limits_M r_1^n P_{nm}(\cos\varphi_1)\cos m\lambda_1\,dm \quad\text{and}\quad d'_{nm}=\int\limits_M r_1^n P_{nm}(\cos\varphi_1)\sin m\lambda_1\,dm$$

are proportional (with an accuracy to constant multiples) to the expression

$$Y_{\alpha\beta\gamma}=\int\limits_M x^\alpha y^\beta z^\gamma\,dm, \tag{III.7}$$

where x, y, z are associated with the spherical coordinates of relationships (I.1'), while α, β, γ are whole numbers, $\alpha + \beta + \gamma$ being equal to n [32].

The quantities $Y_{\alpha\beta\gamma}$ are the moments of mass of the Earth; however, they have a graphic physical meaning only when $n \leq 2$ (the coordinates of the center of inertia and the moments of inertia).

Let us now express c_{nm} and d_{nm} in terms of the moments of inertia of the Earth. It is sufficient for this purpose to consider the various terms of expression (III.3), taking account of formula (III.4). In this regard, (III.3) will be integrated with respect to the elements of mass M with instantaneous coordinates r_1, ϕ_1, λ_1. The order of the function $P_{nm}(\cos\gamma) = P_{nm}(\phi, \lambda)$ does not change after integration, and the coefficients associated with the elementary spherical functions will be the coefficients c_{nm} and d_{nm} in which we are interested.

When n = 0, since $P_0(\cos\gamma) = 1$ (see Appendix II),

$$I_0 = \int_M r_1^0 P_0(\cos\gamma)dm = M. \tag{III.8}$$

Consequently, the coefficient associated with the spherical harmonic of order zero will equal the mass of the Earth.

When n = 1, we have $P_1(\cos\gamma) = \cos\gamma$ (see Appendix II). Therefore, if /249
we convert to rectangular coordinates for simplicity, then with regard to (III.1), we get

$$I_1 = \int_M r_1 P_1(\cos\gamma)dm = \int_M r_1 \cos\gamma\, dm =$$

$$= \frac{1}{r}\int_M (xx_1 + yy_1 + zz_1)dm = \frac{1}{r}\left[x\int_M x_1 dm + y\int_M y_1 dm + z\int_M z_1 dm\right]. \tag{III.9}$$

The integrals

$$MX_c = \int_M x_1 dm; \quad MY_c = \int_M y_1 dm; \quad MZ_c = \int_M z_1 dm \tag{III.10}$$

determine the position of the center of inertia for the mass of the Earth (X_c, Y_c, Z_c) in the selected coordinate system.

If we use formulas (I.1') to convert to spherical coordinates in (III.8), we get (taking account of the form of the spherical functions according to

Appendix II)

$$Y_1 = \int_M r_1 P_1(\cos\gamma)dm = M[Z_c P_{10} + X_c P_{11}\cos\lambda + Y_c P_{11}\sin\lambda]. \qquad \text{(III.11)}$$

The third term in expansion (III.3) contains the integral

$$Y_2 = \int_M r_1^2 P_2(\cos\gamma)dm = \int_M r_1^2\left(\frac{3}{2}\cos^2\gamma - \frac{1}{2}\right)dm = \frac{1}{r^2}\int_M\left[\frac{3}{2}(xx_1+yy_1+zz_1)^2 - \right.$$

$$\left. -\frac{1}{2}r_1^2 r^2\right]dm = \frac{1}{r^2}\int_M\left[\frac{3}{2}(x^2x_1^2+y^2y_1^2+z^2z_1^2)+3(xyx_1y_1+xzx_1z_1+\right.$$

$$\left.+yzy_1z_1)-\frac{1}{2}(x^2+y^2+z^2)(x_1^2+y_1^2+z_1^2)\right]dm = \frac{1}{r^2}\int_M\left[\frac{x^2}{2}(2x_1^2-y_1^2-z_1^2)+\right.$$

$$\left.+\frac{y^2}{2}(2y_1^2-x_1^2-z_1^2)+\frac{z^2}{2}(2z_1^2-x_1^2-y_1^2)+3yzy_1z_1+3xzx_1z_1+3xyx_1y_1\right]dm.$$

If the moments of inertia with respect to the principal axes are designated by

$$A = \int_M (y_1^2+z_1^2)dm; \quad B = \int_M (x_1^2+z_1^2)dm; \quad C = \int_M (x_1^2+y_1^2)dm, \qquad \text{(III.12)}$$

while the centrifugal moments of inertia are given by

$$D = \int_M y_1 z_1 dm; \quad E = \int_M x_1 z_1 dm; \quad F = \int_M x_1 y_1 dm, \qquad \text{(III.12')}$$

then the expression for Y_2 is written /250

$$Y_2 = \frac{1}{r^2}\left[\frac{x^2}{2}(B+C-2A)+\frac{y^2}{2}(A+C-2B)+\frac{z^2}{2}(A+B-2C)+3Dyz+3Exz+\right.$$

$$\left.+3Fxy\right] = \frac{1}{r^2}\left[(A+B-2C)\left(z^2-\frac{1}{2}x^2-\frac{1}{2}y^2\right)+\frac{3}{2}(B-A)(x^2-y^2)+\right.$$

$$\left.+3Dyz+3Exz+3Fxy\right].$$

Converting again to spherical coordinates by formulas (I.1'), we get

$$Y_2 = \frac{1}{r^2}[(A+B-2C)(r^2\cos^2\varphi - \tfrac{1}{2}r^2\sin^2\varphi) + (B-A)r^2\cos^2\varphi(\cos^2\lambda - \sin^2\lambda) +$$
$$+3Dr^2\sin\varphi\cos\varphi\sin\lambda + 3Er^2\sin\varphi\cos\varphi\cos\lambda + 3Fr^2\sin^2\varphi\sin\lambda\cos\lambda].$$

Finally, we have

$$Y_2 = \int_M r_1^2 P_2(\cos\gamma)dm = (A+B-2C)P_{20}(\cos\varphi) + EP_{21}(\cos\varphi)\cos\lambda +$$
$$+DP_{21}(\cos\varphi)\sin\lambda + (B-A)P_{22}(\cos\varphi)\cos 2\lambda + FP_{22}(\cos\varphi)\sin 2\lambda. \qquad \text{(III.13)}$$

Thus, the first coefficients of expansion (2.6) with regard to expression (III.8), (III.11), (III.13) are written in the form

$$c_{00} = 1; \quad c_{10} = \frac{Z_c}{r_0}; \quad c_{11} = \frac{X_c}{r_0}; \quad d_{11} = \frac{Y_c}{r_0};$$

$$c_{20} = \frac{A+B-2C}{2Mr_0^2}; \; c_{21} = \frac{E}{Mr_0^2}; \; d_{21} = \frac{D}{Mr_0^2}; \; c_{22} = \frac{B-A}{4Mr_0^2}; \; d_{22} = \frac{F}{2Mr_0^2}.$$

IV. PARAMETERS OF THE GRAVITATIONAL FIELD OF THE EARTH

1. The values of the coefficients in the expansion for the gravitational potential of the Earth according to I. D. Zhongolovskiy [32] are as follows:

$c_{20} = -109808 \cdot 10^{-8}$;	$d_{32} = -50 \cdot 10^{-8}$;	$c_{42} = 0 \cdot 10^{-8}$;
$c_{22} = 574 \cdot 10^{-8}$;	$c_{33} = 42 \cdot 10^{-8}$;	$d_{42} = 8 \cdot 10^{-8}$;
$d_{22} = -158 \cdot 10^{-8}$;	$d_{33} = 34 \cdot 10^{-8}$;	$c_{43} = 5 \cdot 10^{-8}$;
$c_{30} = 442 \cdot 10^{-8}$;	$c_{40} = 358 \cdot 10^{-8}$;	$d_{43} = -1 \cdot 10^{-8}$;
$c_{31} = 199 \cdot 10^{-8}$;	$c_{41} = -67 \cdot 10^{-8}$;	$c_{44} = 1 \cdot 10^{-8}$;
$d_{31} = -96 \cdot 10^{-8}$;	$d_{41} = -40 \cdot 10^{-8}$;	$d_{44} = 2 \cdot 10^{-8}$;
$c_{32} = 36 \cdot 10^{-8}$;		

$r_0 = 6\ 363\ 553$ м; $\mu = fM = 398\ 590 \cdot 10^9$ м^3 sec^{-2}.

2. The values of the constants accepted at the International Astronomical Union in 1964 (Hamburg) are: /251

$$r_0 = 6\,378\,160\,\mathrm{m};\quad \mu = fM = 398\,603 \cdot 10^9\,\mathrm{m}^3\,\mathrm{sec}^{-2};\quad \alpha = 1:298.25.$$

3. The values of the normalized coefficients in the expansion for the gravitational potential determined from satellite data by non-Soviet research workers Izhik and Kaula from analysis of optically observed satellites, Guire from analysis of radio measurements of Doppler velocities are listed in Table 39.

TABLE 39 *

Normalized Coefficient	Izhik 1964	Kaula 1963	Guire 1964
$\bar{c}_{20} \cdot 10^6$	-	-484,08	-
$\bar{c}_{22} \cdot 10^6$	1,17	1,88±0,29 [1]	2,60
$\bar{d}_{22} \cdot 10^6$	-0,95	-1,38±0,17	-0,99
$\bar{c}_{30} \cdot 10^6$	-	0,97±0,01	-
$\bar{c}_{31} \cdot 10^6$	0,81	1,52±0,03	1,64
$\bar{d}_{31} \cdot 10^6$	-0,25	0,14±0,16	0,18
$\bar{c}_{32} \cdot 10^6$	0,24	-0,02±0,26	0,84
$\bar{d}_{32} \cdot 10^6$	-0,25	0,42±0,06	-0,07
$\bar{c}_{33} \cdot 10^6$	-0,50	0,70±0,26	1,08
$\bar{d}_{33} \cdot 10^6$	0,93	0,76±0,29	1,01
$\bar{c}_{40} \cdot 10^6$	-	0,67±0,02	-
$\bar{c}_{41} \cdot 10^6$	-0,18	-0,33±0,01	-0,80
$\bar{d}_{41} \cdot 10^6$	-0,25	0,37±0,15	-0,49
$\bar{c}_{42} \cdot 10^6$	-0,11	0,01±0,02	0,27
$\bar{d}_{42} \cdot 10^6$	0,23	0,35±0,015	1,19
$\bar{c}_{43} \cdot 10^6$	0,28	0,17±0,02	1,33
$\bar{d}_{43} \cdot 10^6$	-0,08	0,41±0,03	-0,05
$\bar{c}_{44} \cdot 10^6$	-0,08	-0,01±0,08	-0,37
$\bar{d}_{44} \cdot 10^6$	0,29	0,18±0,05	0,31

4. The values of the coefficients associated with the zonal harmonics in the expansion for the gravitational potential (Jeffreys coefficients) determined by non-Soviet researchers [95, 96]

$$I_n = -\sqrt{2n+1}\,\bar{c}_{n0},$$

[1] Mean square error.

*Tr. Note: Commas indicate decimal points.

where $\bar{c}_{n0}$ is the normalized coefficient, are given in Table 40.

TABLE 40 * /252

Coefficients	King-Hele, Cook 1964	Kozai 1964
$I_2 \cdot 10^6$	1082,70	1082,65
$I_3 \cdot 10^6$	–	–2,53
$I_4 \cdot 10^6$	–1,40	–1,62
$I_5 \cdot 10^6$	–	–0,21
$I_6 \cdot 10^6$	0,37	0,61
$I_7 \cdot 10^6$	–	–0,32
$I_8 \cdot 10^6$	0,07	–0,24
$I_9 \cdot 10^6$	–	–0,10
$I_{10} \cdot 10^6$	–0,50	–0,10
$I_{11} \cdot 10^6$	–	0,28
$I_{12} \cdot 10^6$	0,31	–0,28
$I_{13} \cdot 10^6$	–	–0,18
$I_{14} \cdot 10^6$		0,19

*Tr. Note: Commas indicate decimal points.

V. DERIVATION OF EQUATIONS IN OSCULATING ELEMENTS WITH RESPECT TO COMPONENTS q AND k OF THE LAPLACE VECTOR

The differential equations of disturbed motion with respect to components q and k of the Laplace vector may be derived in various ways. We shall find them according to the general method used in celestial mechanics directly from the expressions which relate the kinematic characteristics of Keplerian motion to its first integrals without resorting at all to the concepts of orbital eccentricity e and angular distance of the perigee ω[1]. In this connection, we shall use the basic rule [1, 2] according to which equations in osculating elements are derived by differentiation of the first integrals of undisturbed motion with respect to time. In this type of differentiation, time t and the coordinates are treated as constants, and the derivatives of the velocity components are replaced by the components of the disturbing acceleration.

Thus equations (6.3) are the initial relationships in the following derivation.

In order to derive the equation with respect to dq/dt, we shall express the functions sin u and cos u in the first of the relationships in (6.3) in terms of the inertial rectangular coordinates x, y, z by means of (6.8'). This must be done since the argument of latitude in the case of differentiation with respect to time is a disturbed function and cannot be taken as constant (according to the rule cited above) like the ordinary coordinates.

After eliminating sin u and cos u, we get

$$q=\sqrt{\frac{p}{\mu}}\,V_r\,\frac{z}{r}\,\frac{1}{\sin i}+\left(\sqrt{\frac{p}{\mu}}\,V_\tau-1\right)\left(\frac{x}{r}\cos\Omega+\frac{y}{r}\sin\Omega\right).$$

Differentiating this equality with respect to time, according to the given rule, we get

---

[1] As is known [1, 2, 48], e and ω are expressed in terms of components f_1 and f_2 of the Laplace vector in the absolute geocentric rectangular coordinate system. In this case

$$e=\frac{1}{\mu}\sqrt{f_1^2+f_2^2}=\sqrt{q^2+k^2};\quad \mathrm{tg}\,\omega=\frac{f_2}{f_1}.$$

$$\frac{dq}{dt}=\frac{1}{2\sqrt{\mu p}}V_r\frac{z}{r}\frac{1}{\sin i}\frac{dp}{dt}+\sqrt{\frac{p}{\mu}}\frac{z}{r}\frac{1}{\sin i}S-\sqrt{\frac{p}{\mu}}V_r\frac{z}{r}\frac{\cos i}{\sin^2 i}\frac{di}{dt}+$$

$$+\left(\frac{1}{2\sqrt{\mu p}}V_\tau\frac{dp}{dt}+\sqrt{\frac{p}{\mu}}T\right)\left(\frac{x}{r}\cos\Omega+\frac{y}{r}\sin\Omega\right)+\left(\sqrt{\frac{p}{\mu}}V_\tau-1\right)\left(-\frac{x}{r}\sin\Omega+\frac{y}{r}\cos\Omega\right)\frac{d\Omega}{dt}.$$

By reverse substitution, we eliminate coordinates x, y, z, and after reducing similar terms, we get

$$\frac{dq}{dt}=\frac{1}{2p}(q+\cos u)\frac{dp}{dt}+\left(\frac{1}{2}k\sin 2u-q\sin^2 u\right)\operatorname{ctg} i\frac{di}{dt}+$$

$$+\left(\frac{1}{2}q\sin 2u+k\sin^2 u\right)\cos i\frac{d\Omega}{dt}+\tilde{S}\sin u+\tilde{T}\cos u.$$

The functions $\tilde{S}$ and $\tilde{T}$ here are the same as in (6.4').

We may eliminate the derivatives dp/dt, di/dt and dΩ/ dt from this equation by using the equations of (6.4).

The equation with respect to dk/dt is derived in the same manner. The second relationship in (6.3) serves as the initial expression in this case.

VI. DISTURBANCES OF THE ORBIT OF A CIRCULAR SATELLITE IN THE FIELD OF THE SPHEROIDAL EARTH

A circular satellite is defined as one whose velocity vector at the initial instant corresponds to circular Keplerian motion.

Some analysis of the disturbed orbits of circular satellites carried out on the basis of numerical calculations is given in §8. In the given appendix, this problem is studied in more detail with the aid of the approximate analytical solutions derived in §11. These solutions describe disturbances of Keplerian orbital elements in the field of the terrestrial spheroid (the concept of a spheroid is defined in §6) with an accuracy of second-order polar oblateness of the Earth.

In order to study the perturbations of the elements of a circular satellite, we should set $q_0 = k_0 = 0$ in the solutions found in §11. The resultant relationships show that disturbance of the plane and of the focal parameter of the orbits of circular satellites is in no way different in principle from the disturbances of these same elements for satellites with an elliptical orbit. It is therefore most interesting to examine the changes in shape and position of an osculating orbit which are described by the relationships q(u) and k(u). When $q_0 = k_0 = 0$, these functions contain periodic (short-period) and quasisecular parts. The quasisecular part is linearly dependent on the argument of latitude u, changes extremely slowly (in proportion to ε^2) and is introduced into equations (11.9) from the second approximation (F_{q2} and F_{k2}). We shall limit ourselves initially to studying equations of the first approximation. In this case, q and k are periodic functions and are represented by the equations: /255

$$\left.\begin{aligned} q &= \varepsilon(\cos u-\cos u_0)[a_1-a_2(\cos^2 u+\cos u\cos u_0+\cos^2 u_0)];\\ k &= \varepsilon(\sin u-\sin u_0)[b_1+b_2(\sin^2 u+\sin u\sin u_0+\sin^2 u_0)].\end{aligned}\right\} \quad \text{(VI.1)}$$

$$\left.\begin{aligned} a_1 &= \frac{1}{2}(3\sin^2 i_0-1);\quad b_1=-\frac{1}{2};\\ a_2 &= b_2=\frac{7}{6}\sin^2 i_0.\end{aligned}\right\} \quad \text{(VI.2)}$$

In the paragraphs which follow, we shall investigate the effect which the quasilinear term--the most significant component of the second-approximation functions F_{q2} and F_{k2} -- has on the qualitative picture described by equations (VI.1).

Each of the functions (VI.1) has six complex or real zeros in the interval $0 \leq u \leq 2\pi$, and the arrangement of these zeros determines the properties of the osculating orbit. Let us examine this problem in more detail.

One group of zeros depends only on the value of u_0, since it is found from the conditions

$$\left.\begin{aligned} \sin u &= \sin u_0;\\ \cos u &= \cos u_0.\end{aligned}\right\} \quad \text{(VI.3)}$$

A second group depends on the two parameters i_0 and u_0, and comprises the zeros of the trinomials enclosed in brackets in (VI.1).

Functions (VI.1) are a two-parameter family which depends on the initial values of i_0 and u_0.

Let us set $u_0 = 0$ (unless otherwise stated, this is assumed everywhere in the following discussion) and find the zeros of the expressions

$$\left.\begin{aligned}\tilde{q} &= a_1 - a_2(\cos^2 u + \cos u + 1);\\ \tilde{k} &= b_1 + b_2 \sin^2 u.\end{aligned}\right\} \tag{VI.1'}$$

Setting $\tilde{q}$ equal to zero, we shall solve the resultant equation with respect to cos u. /256

We get

$$\cos u = -\frac{1}{2} \pm \frac{1}{2}\sqrt{4\frac{a_1}{a_2} - 3}. \tag{VI.3'}$$

Since cos u should assume only real values, the condition

$$4\frac{a_1}{a_2} - 3 \geq 0,$$

should be fulfilled, and hence we get (with regard to (VI.2))

$$\sin^2 i_0 \geq 4/5$$

or

$$i_0 \geq i_{0_1} = \arcsin 0{,}89\dot{2} = 63°10'.$$

Consequently, the function $\tilde{q}$ can have zeros only when $i_0 \geq i_{0_1}$. If $i_0 = i_{0_1}$, we get for cos u from (VI.3') only the single value cos u = -1/2, i.e. at this inclination the function $\tilde{q}$ has multiple zeros. In the case $i_0 > i_{0_1}$, the function $\tilde{q}(u)$ (considered with respect to argument u) has four different zeros. For these zeros when $i_0 = \pi/2$, we get the following values of the argument

$$\left.\begin{aligned} u_{1,2} &= 180^\circ \pm 80^\circ 52'; \\ u_{3,4} &= 180^\circ \pm 33^\circ 55'. \end{aligned}\right\} \tag{VI.3''}$$

As i_0 decreases in the range [90°, 63°,10'], zeros u_1, u_3 and u_2, u_4 converge by pairs.

Investigating the second function in (VI.1'), we find that the natural requirement $\sin^2 u \leq 1$ is fulfilled (with regard to the second expression in (VI.1') and (VI.2)) only at

$$i_0 \geq i_{0_2} = \arcsin 0{,}658 = 41^\circ 08'.$$

Consequently, the function $\tilde{k}(u)$ has zeros only when $i_0 \geq i_{0_2}$.

Setting this function equal to zero, we get the equality

$$\sin^2 u = 3/(7 \sin^2 i_0),$$

from which it follows that when $i_0 = \pi/2$, the function $\tilde{k}(u)$ has four different zeros at the points

$$\left.\begin{aligned} u_{5,6} &= 90^\circ \pm 41^\circ 08'; \\ u_{7,8} &= 270^\circ \pm 41^\circ 08'. \end{aligned}\right\} \tag{VI.3'''}$$

As i_0 is reduced in the range [90°, 41°08'], zeros u_5, u_6 and u_7, u_8 converge, merging at the end of the interval (where $i_0 = i_{02}$). At this point, the function $\tilde{k}(u)$ has two multiple zeros. Let us find out whether equations (VI.1') have common zeros. For this purpose, we solve them simultaneously /257

and find that when $i_0 = i_{0_k} = \arcsin \sqrt{7/8} = 70^\circ 03'$, the zeros of functions $\tilde{q}(u)$ and $\tilde{k}(u)$ coincide. This takes place at values of the argument $u = 180^\circ \pm$ $\pm\ 44^\circ 24' 50''$.

Thus, the arrangement of the real zeros of functions (VI.1) for $u_0 = 0$ and various inclinations of the orbit $0 < i_0 < \pi$ are illustrated by the graph in Fig. 118. Three characteristic inclinations may be pointed out (since the zeros are located in plane (i_0, u) symmetric to the line $i_0 = \pi/2$, we shall consider only the first quadrant $0 < i_0 \leq \pi/2$).

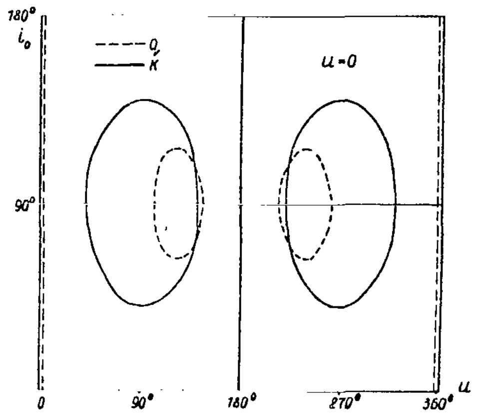


Fig. 118

In the first of them ($i_{0_k} = \arcsin \sqrt{7/8} = 70°03'$), both equations in (VI.1) have common zeros. This point (i_0, u) may be called the critical point in view of certain properties which will be clear from the following discussion.

At the second characteristic inclination ($i_{0_1} = \arcsin 0.892 = 63°10'$), multiple zeros appear in the first equation of (VI.1), and at the third characteristic inclination ($i_{0_3} = \arcsin 0.658 = 41°08'$), multiple zeros appear in the second equation.

Treating the argument of latitude u in equation (VI.1) as a parameter, we get a curve in plane (q, k) which is the hodograph of the normalized Laplace vector[1]. Let us call this hodograph, which corresponds to the value $u_0 = 0$, the zero hodograph. The zero hodographs for the three charactertistic inclinations, as well as for the values $i_0 = 90°$, $i_0 = 30°$ and $i_0 = 5°$ are shown in Figs. 119-122; the scale for argument u is given on the curve. According to the construction, the modulus of the radius vector for any point on the hodograph is equal to the eccentricity at the given value of argument u, and the angle between the radius vector and axis q is equal to the angular distance of the perigee ω.

As these curves show, the zero hodograph is always symmetric with respect to the q-axis and passes through the coordinate origin. When $i_0 = \pi/2$, the hodograph is an extremely complex curve with several nodes located in all quadrants of plane (q, k) and looping the coordinate origin several times.

[1] That is, the Laplace vector divided by the constant $\mu = fM$. /257

As the inclination is reduced, the zero hodograph is shifted entirely into the left half-plane and "straightens out", tending to the limit of a circle as $i_0 \to 0$. /259

/258

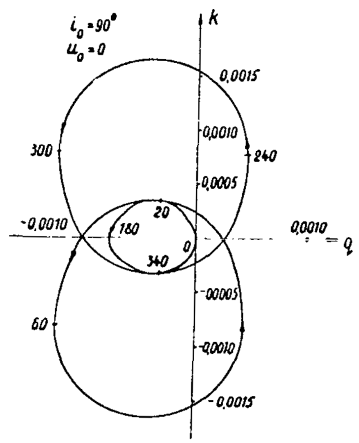


Fig. 119

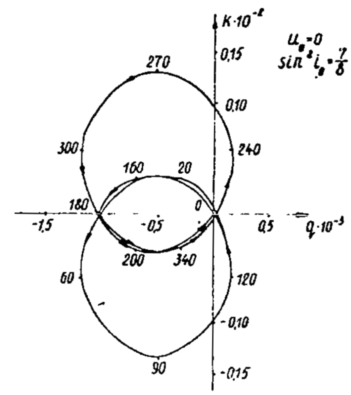


Fig. 120

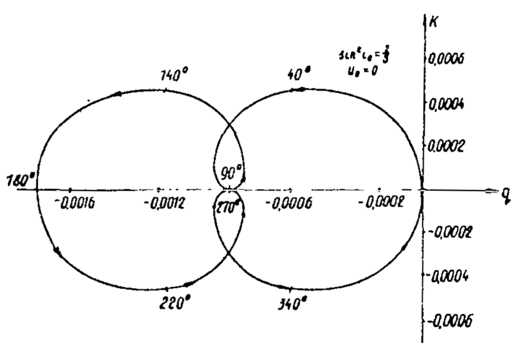


Fig. 121

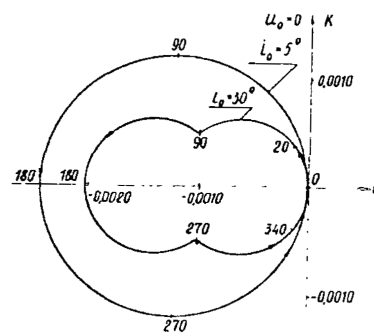


Fig. 122

Analyzing the arrangement of the zeros for the functions

$$\left.\begin{array}{l} q(i_0, u); \\ k(i_0, u), \end{array}\right\} \qquad \text{(VI.4)}$$

we find characteristic differences in the osculating orbits of circular satellites of the spheroidal Earth.

Let us first determine the form of the osculating orbit.

The instantaneous value of eccentricity is given by the equation

$$e = \sqrt{q^2 + k^2}.$$

The equality e = 0 corresponds to coincidence of the zeros of functions (VI.4), i.e. to the common points of the curves shown in Fig. 118.

According to (VI.3), one group of common zeros takes place at $u = 2n\pi$ ($n = 0, 1, 2\ldots$) and at any inclinations i_0[1]. Another group (when $2n\pi < u < 2(n+1)\pi$) is found as a result of examining the system of equations

$$\left.\begin{aligned} a_2(i_0)[\cos^2 u + \cos u + 1] &= a_1(i_0); \\ b_2(i_0)\sin^2 u &= 1/2, \end{aligned}\right\}$$

which is obtained from the expressions in (VI.1) enclosed in brackets.

The real solutions are

$$\begin{aligned} u^{*} &= (2n+1)\pi + 44^\circ 1',5; \\ u^{**} &= (2n+1)\pi - 44^\circ 1',5, \quad n = 0, 1, 2, \ldots, \\ i_{0_k} &= \frac{\pi}{2} \pm 19^\circ 57'. \end{aligned}$$

which define critical inclination i_0 and the values of the argument of latitude u at which the eccentricity within the period vanishes.

Hence, it follows that the osculating orbit of a circular satellite of the spheroidal Earth is an ellipse with eccentricity which vanishes at the points $u = 2n\pi$ ($n = 0, 1, 2$) for any initial orbitral inclinations. In addition, when the inclination is equal to the critical value ($i_{0_k} = \pi/2 \pm 19^\circ 57'$), the eccentricity also vanishes at the points $u = (2n + 1)\pi \pm 44^\circ 24' 50''$.

This property of the orbit of a circular satellite is illustrated by /260

[1] Generally speaking, according to (VI.3) common zeros are realized at any initial conditions i_0, u_0 and values of the argument $u = u_0 + 2n\pi$. /259

Figures 119 - 122, where vanishing of the eccentricity corresponds to passage of the hodograph through the coordinate origin.

Let us examine the behavior of eccentricity in the neighborhood of points where $e = 0$. For this purpose, we expand sin u and cos u in equations (VI.1) in a Taylor series in the neighborhood of the point $u = u_0$. Limiting ourselves to terms of the first negative order of magnitude with respect to $\Delta u = u - u_0$, we get

$$\left.\begin{aligned} q &= \varepsilon(-\Delta u \sin u_0)(a_1 - a_2 \cos^2 u_0); \\ k &= \varepsilon \Delta u \cos u_0 (b_1 + b_2 \sin^2 u_0). \end{aligned}\right\} \tag{VI.5}$$

Hence, taking into consideration the values of the coefficients in (VI.2), we have when $u_0 = 0$

$$e = 0.5\,|\varepsilon|\,|\Delta u|, \tag{VI.6}$$

and when $u_0 = u^* = \pi + 44°1.5'$ or when $u_0 = u^{**} = \pi - 44°1.5'$

$$e = 1.38\,|\varepsilon|\,|\Delta u|. \tag{VI.7}$$

On the basis of (VI.6) and (VI.7), we conclude that in the neighborhood of points $u = 2n\pi$ as well as $u = u^*$ and $u = u^{**}$ (at the critical inclination), the eccentricity of the osculating orbit of a circular satellite is an infinitesimal of the same order as the increment in the argument of latitude.

Keeping this situation in mind, let us go on to determining the position of the osculating orbit. Let us first examine the change in the angular distance of the perigee in the neighborhood of those points where eccentricity vanishes.

Since the hodograph of the Laplace vector is a smooth curve, there should be a discontinuity in the function $\omega(u)$ when this curve intersects the coordinate origin. If it is assumed that $\Delta u < 0_2$, then on the basis of (VI.5) we may write in the left-hand neighborhood of the point $u = u_0$

$$\left.\begin{aligned} q &= \varepsilon \Delta u \sin u_0 (a_1 - a_2 \cos^2 u_0); \\ k &= \varepsilon \Delta u \cos u_0 (-b_1 - b_2 \sin^2 u_0). \end{aligned}\right\} \tag{VI.8}$$

If we now assume $u_0 = 2n\pi$ in (VI.5) and (VI.8) and divide the left-hand and right-hand members of these equalities by (VI.6), then taking ε as negative we get when $u = 2n\pi$: from the right--$\cos \omega = 0$, $\sin \omega = 1$, i.e. $\omega = \pi/2$, and from the left-- $\cos \omega = 0$, $\sin \omega = -1$, i.e. $\omega = 3\pi/2$.

If we now assume $u_0 = u^*$ or $u_0 = u^{**}$ (VI.5) and (VI.8) and we divide these equalities by (VI.7), we may perform a similar analysis for the critical points u^* and u^{**}.

In this way we find that the angular distance of the perigee ω in the neighborhood of the points $u = 2n\pi$, u^*, u^{**} has a discontinuity of the first kind equal to π. At the points $u = 2n\pi$, $\omega = \pi/2$ from the right and $3\pi/2$ from the left, at the points $u = u^*$, $\omega = \pi$ from the right and zero from the left; at points $u = u^{**}$, $\omega = 0$ from the right and π from the left. /261

Hence, it follows in particular, that the perigee comes at the point $\omega = \pi/2$ at a moment infinitesimally close to the initial time ($u = +0$)[1].

From the hodographs (see Fig. 119 - 122) it is obvious that the perigee is moving in the forward direction at the initial instant. In order to give an analytical demonstration of this, we write the expansions of $\sin u$ and $\cos u$, retaining terms to $(\Delta u)^2$ inclusive. Let us substitute them in (VI.1) and divide both members of the equations by (VI.6). If we then assume $u_0 = 0$, then in the neighborhood of this point we get

$$\left.\begin{aligned} &\sin\omega \simeq \pm 1; \\ &\cos\omega = \Delta u(a_1 - 3a_2). \end{aligned}\right\} \tag{VI.9}$$

On the interval $0 < i_0 \leq \pi/2$, the condition

$$a_1 - 3a_2 < 0. \tag{VI.10}$$

is satisfied. This follows from the obvious inequalities

$$\frac{1}{2}\left(3\sin^2 i_0 - 1\right) - \frac{7}{2}\sin^2 i_0 < 0; \quad -\frac{4}{2}\sin^2 i_0 - \frac{1}{2} < 0.$$

[1] Here and in the following discussion the signs + and - will be used to denote the right-hand and left-hand neighborhoods of a point, respectively.

Therefore, it follows from (VI.9) that at an instant close to the initial time, $\sin \omega > 0$, $\cos \omega < 0$, i.e. the angular distance of the perigee changes in the second quadrant, and since $\omega = \pi/2$ at the initial instant, the perigee of the osculating orbit is moving in the forward direction in the right-hand neighborhood of the point $u = u_0 = +0$, in other words, the angle ω is increasing.

Let us now determine the nature of motion of the line of apsides.

This may be done simply by examining the zeros of the functions q(u) and k(u) in Fig. 118. Let us write these functions in the form

$$\left.\begin{aligned} q(u) &= \varepsilon(\cos u - 1)q^*(u); \\ k(u) &= \varepsilon \sin u\, k^*(u). \end{aligned}\right\} \tag{VI.11}$$

Then with regard to (VI.3") and (VI.3'''), function q(u) will have zeros when $i_0 = \pi/2$ at values of the argument $u_{1q} = 0°$, $u_{2q} = 99°8'$, $u_{3q} = 146°5'$, $u_{4q} = 213°55'$, $u_{5q} = 260°52'$, $u_{6q} = 360°$, while function k(u) will have zeros for the same i_0 at $u_{1k} = 0°$, $u_{2k} = 48°52'$, $u_{3k} = 131°8'$, $u_{4k} = 180°$, $u_{5k} =$ /262
$= 228°52'$ and $u_{6k} = 311°8'$.

As may be seen, the zeros of q(u) and k(u) alternate in the interval $0 < u < 360°$, which indicates that the line of apsides rotates at inclination $i_0 = \pi/2$. As was pointed out above, this rotation takes place in the forward direction and corresponds in Fig. 119 to the fact that the origin of the coordinate system (q, k) is located within the loops formed by the hodograph for the Laplace vector.

The unequal distances between the zeros along the u-axis (see Fig. 118) indicates nonuniform rotation, while the number of zeros shows that the line of apsides makes 2.5 revolutions in a single draconic period of revolution of the satellite. Actually, each of the functions q or k becomes zero twice for one revolution of the line of apsides. However, although q and k have six zeros apiece, the line of apsides makes not 3, but 2.5 revolutions since one of the zeros corresponds to the initial position of the satellite.

As the inclination of the orbit decreases, the zeros of q(u) and k(u) converge by pairs, i.e. nonuniformity of rotation increases, and finally, when the zeros merge at the critical inclination, rotation of the line of apsides is replaced by oscillation.

This, then, is the part played by the critical inclination: it divides the region of rotation of the line of apsides from the region of oscillation, and also the eccentricity of the osculating orbit vanishes in mid-period at this inclination.

Past the critical inclination, the zeros no longer alternate (see Fig. 118), and over the entire remaining range of inclinations ($0 < i_0 < i_{0_{cr}}$) the line of apsides oscillates only. After each of the characteristic inclinations (63°10' and 41°8') is passed, the number of real zeros for the function q(u) and then for k(u) also, decreases by two. In each of these cases, the oscillations are limited first to 180°, and then to 90° (with respect to the motion resulting from the fact that the perigee starts at $\omega = \pi/2$ at the beginning of a revolution, and ends at $\omega = 3\pi/2$ at the end of a revolution). All of these phenomena are readily apparent in Figs. 119 - 122.

Let us now change the initial conditions, assuming $u_0 \neq 0$. For convenience, we write equation (VI.1) in the form

$$\left.\begin{aligned} q &= \varepsilon f_1(u, i_0) - \varepsilon f_1(u_0, i_0); \\ k &= \varepsilon f_2(u, i_0) - \varepsilon f_2(u_0, i_0). \end{aligned}\right\} \qquad \text{(VI.12)}$$

At each fixed value of the initial conditions, the functions $f_1(u_0, i_0) =$ = const and $f_2(u_0, i_0)$ = const. Therefore, the condition $u_0 \neq 0$ results in parallel displacement of the zero hodograph in plane (q, k) and in a change in the scale of the graph. The form of the hodograph itself does not change with a variation in the initial value of the argument of latitude. /263

If we use the notation

$$q = q(u, i_0, 0); \quad k = k(u, i_0, 0)$$

and

$$\tilde{q} = q(u, i_0, u_0); \quad \tilde{k} = k(u, i_0, u_0),$$

then according to (VI.12), displacement of the hodograph in plane (q, k) with a change in u_0 is equal to

$$\Delta q = \tilde{q} - q = \varepsilon f_1(u_0, i_0) - \varepsilon f_1(0, i_0);$$
$$\Delta k = \tilde{k} - k = \varepsilon f_2(u_0, i_0) - \varepsilon f_2(0, i_0),$$

i.e. it is described by the zero hodograph in which the argument u is replaced by u_0.

Thus, when the initial value of the argument $u_0 \neq 0$, each point of the zero hodograph will move in its own plane by the quantities Δq and Δk equal respectively, to the abscissa and ordinate of the point $u = u_0$ of the zero hodograph taken at the same value of inclination i_0.

Consequently, if we know the zero hodograph, we may construct the hodograph of the normalized Laplace vector for any value of the argument u_0, and we may also find the displacement of the hodograph with a transition from one value $u_0 \neq 0$ to another. The properties of the osculating orbit are determined not only by the form of the hodograph, but by the positions of the points where it intersects the coordinate axes (i.e. these properties are determined by the zeros of functions (VI.1)). Therefore, in view of the nonlinearity of f_1 and f_2, we cannot say beforehand how the orbit will behave at various values $u_0 \neq 0$. In particular, it has already been pointed out above that the eccentricity of an osculating orbit always vanishes at the points $u = u_0 + 2n\pi$.

In order to explain the effect of the quasisecular part, let us write the equations for q and k with regard to this term in the form

$$\left.\begin{aligned} q &= \varepsilon f_1(u, i_0) + \varepsilon[2(n-1)Q_1\varepsilon\pi - f_1(u_0, i_0)] + \varepsilon^2 Q_1 u; \\ k &= \varepsilon f_2(u, i_0) + \varepsilon[2(n-1)K_1\varepsilon\pi - f_2(u_0, i_0)] + \varepsilon^2 K_1 u, \end{aligned}\right\} \quad \text{(VI.13)}$$

Here, K_1 and Q_1 designate the coefficients associated with quasisecular terms, n is the number of the orbit on which satellite motion is being considered, /264 and u varies over the range $[0, 2\pi]$.

As may be seen from (VI.13) the presence of a quasisecular part in the equations has a double effect.

In the first place, the effect of the quasisecular term, just as the effect of the initial argument $u_0 \neq 0$, is to shift the hodograph in the plane, in the given case by a quantity proportional to the number of preceding revolutions[1]

Thus, accounting for this part of the quasisecular functions results in a shift of the zero hodograph with respect to axes q and k, this shift being equal to

$$\Delta q = 2\varepsilon^2 (n-1) Q_1 \pi; \quad \Delta k = 2\varepsilon^2 (n-1) K_1 \pi.$$

In the second place, the effect of the quasisecular part shows up in the form of a linear term with an extremely small coefficient and leads as it were to continuous deformation of the hodograph, i.e. to a certain change in the properties of the orbit.

Deformation of the hodograph due to the presence of the quasisecular component is extremely slight. For instance, on the first three revolutions in the case of orbits with a radius of 7,400 km, only the seventh decimal place changes. The effect of quasisecular terms on the zero hodograph decreases as the inclination increases.

One of the characteristic singularities of the osculating orbit of a circular satellite is that its eccentricity may vanish at certain moments of motion. This singularity is also inherent in the orbits of satellites with a fairly small initial eccentricity unequal to zero.

A necessary (but not sufficient) condition for disappearance of eccentricity is

$$|\delta e| \geq e_0, \tag{VI.14}$$

where δe are perturbations of eccentricity and e_0 is the initial value of eccentricity.

[1] In spite of the form of equations (VI.13), the functions q and k will not increase without bound as $n \to \infty$. The Poincaré method may be applied only over a limited range of variation in the argument. As $n \to \infty$, the effect of long-period oscillations in q and k should be felt, and these oscillations are not reflected in the equations since the corresponding harmonics are approximated by a quasisecular term over a small range of variation in u.

Consequently, a satellite may be called nearly circular (or near-circular) when it has an initial eccentricity which is no greater than the maximum absolute values of perturbations in eccentricity.

If it is remembered that $e_0 = \sqrt{q_0^2 + k_0^2}$, a and /265

$$\delta e = \sqrt{q_0^2 + k_0^2 + \varepsilon(F_q^2 + F_k^2) + 2\varepsilon(q_0 F_q + k_0 F_k)} - \sqrt{q_0^2 + k_0^2},$$

we find that condition (VI.14) is satisfied (i.e. the satellite will be nearly circular) if the initial values of q_0 and k_0 lie within the rectangle

$$\left.\begin{aligned} -0.778|\varepsilon| < q_0 < 1.496|\varepsilon|; \\ -0.487|\varepsilon| < k_0 < 0.936|\varepsilon|. \end{aligned}\right\} \qquad \text{(VI.15)}$$

The asymmetry of this rectangle with respect to the axes ($\vec{q}$, $\vec{k}$) is explained by the asymmetry of the zero hodograph with respect to these axes (see Fig. 119 - 122).

As may be seen from (VI.15), the quantities q_0 and k_0 for a nearly circular satellite have the order of oblateness. Therefore the functions $q(u)$ and $k(u)$ are written in the form

$$\left.\begin{aligned} q(u) = \varepsilon[f_1(u, i_0) - f_1(u_0, i_0) + C_1]; \\ k(u) = \varepsilon[f_2(u, i_0) - f_2(u_0, i_0) + C_2], \end{aligned}\right\} \qquad \text{(VI.16)}$$

where $C_1 \lesseqgtr 0$ and $C_2 \lesseqgtr 0$ are constants approximately equal to unity.

Thus, in the case of a nearly circular satellite, the zero hodograph is somewhat shifted in its own plane, while the change in the angular distance of the perigee and eccentricity of an osculating orbit will be determined by the zeros of functions (VI.16).

Let us examine motion of the line of aspides of nearly circular satellites. Since no more than two different branches of the curve intersect at the nodes of a zero hodograph (see Fig. 119- 122), the eccentricity of the osculating orbit of a nearly circular satellite may become zero no more than twice (when the origin of the coordinate system (q, k) coincides with the node of the hodograph). Rotary motion of the line of aspides in these cases alternates (during a single revolution of the satellite) with oscillatory motion. Oscillatory motion may be superimposed on rotational motion in the case where some branch of the hodograph passes through the coordinate origin (e.g. in the case where $i_0 = 90°$ and the point of the zero hodograph corresponding to $u_0 = 30°$ coincides with the coordinate origin). However, pure rotational motion of the line of aspides may be observed. This takes place when the coordinate origin coincides with the point corresponding to $u = 180°$ at inclinations $71° < i_0 \leq 90°$ (see Fig. 119), or when the coordinate source is located somewhere within the central loop of the hodograph at $41°08' < i_0 < 90°$.

Since the functions q and k have no more than six real zeros, by repeating the procedure used above we conclude that in the first case the line of apsides makes 2.5 revolutions, and in the second case an even 3 revolutions per revolution of the satellite. /266

Thus, we may make the following final conclusion: the line of apsides of a nearly circular satellite may go through purely rotational motions at a rate of only 2.5 or only 3 revolutions per satellite revolution. In the first case the initial inclination of the orbit must lie within the range $71° < i_0 < 90°$, while the eccentricity of an osculating orbit becomes zero within the period at $u_0 = 180°$.

In the second case, the initial inclination must lie within the range $41°08' < i_0 \leq 90°$, while the eccentricity of the osculating orbit is nowhere equal to zero.

The line of aspides may also go through other types of motion: oscillatory and rotational (in the course of a single revolution of the satellite), rotational motion with superimposed oscillatory motion, and purely oscillatory motion. In the first case the eccentricity of the osculating orbit vanishes twice in the course of a single revolution of the satellite, while in the second case it vanishes once.

In all cases where the eccentricity vanishes, the line of apsides goes through a jump equal to π.

It should be stated that the facts outlined in this appendix are not an organic singularity of the orbits of satellites with low and zero initial eccentricities. They arise only as a consequence of the specific method of describing motion with the aid of osculating parameters: eccentricity and angular distance of the perigee. And they will always arise when the motion parameters include a quantity which is in some way related to the position of the line of aspides.

Various systems of coordinates or parameters may be selected for describing the motion of satellites without any singularities throughout the entire range of eccentricities. For instance, examples of such systems are the system of phase variables in the rectangular inertial geocentric coordinate system or the system of osculating elements in which the angular distance of the perigee and eccentricity are replaced by two components of the Laplace vector. However, the motion of circular satellites may be made geometrically more graphic by studying the changes in angular distance of the perigee and eccentricity.

VII. EXTREMUM POSITIONS OF THE CIRCULAR ORBITS OF SATELLITES IN THE FIELD OF THE SPHEROIDAL EARTH /267

The perturbations due to eccentricity of the Earth's gravitational field cause only slight changes in the regularities of Keplerian motion. Therefore, a satellite with initial parameters corresponding to an elliptical orbit will be at a minimum or maximum distance from the Earth no more than once during a revolution, namely at those times when it passes through the point of the apogee and perigee of the osculating trajectory. In nearly circular satellites (as a consequence of the similarity between the magnitude of the disturbances of eccentricity and its initial value), this rule may be broken. A circular satellite in particular, due to rotation of the line of apsides, will go through the position of the osculating perigee several times in a single revolution, and consequently will be at minimum distances from the Earth on several occasions. Thus, the problem of determining extremum positions for circular and nearly circular satellites is not trivial.

Let us define this concept more specifically. The extremum positions of the satellite we shall call those in which a local minimum or maximum is reached in the focal radius r. In this case, the necessary condition

$$r' = 0 \qquad \text{(VII.1)}$$

is realized. Condition (VII.1) corresponds to the satellite positions during crossing of the perigee, the apogee of an osculating orbit, and the point where the osculating eccentricity vanishes. We shall call the apogee the furthest removed position from the Earth, and the perigee the least removed distance from the Earth, regardless of the true position of the satellite with respect to the line of apsides and the magnitude of the osculating eccentricity at this moment. In this connection, it should be borne in mind that due to disturbances of the focal parameter (apart from disturbances of the eccentricity and the angular distance of the perigee), the passage of the apogee by a circular satellite in osculating motion does not always correspond to the apogee position (the same applies to the osculating perigee and the perigee position). Therefore, a satellite located at the furthest removed distance (i.e. in the apogee position) will also be at the perigee of the osculating orbit.

There are two possible approaches to the problem as formulated: direct investigation of function (VII.0) and investigation of osculating motion. /268 We shall give both of these approaches, beginning with the latter.

And so, let us determine the extremum positions of a satellite, taking as an example the initial value $u_0 = 0$ and expressing the parameters q and k in terms of the elements of the osculating ellipse e and ω by means of the relationships

$$q = e\cos\omega, \quad k = e\sin\omega.$$

The apogee and perigee positions are determined by the conditions $\omega = u$ (the satellite located at the osculating perigee) and $\omega + \pi = u$ (the satellite located at the osculating apogee), since the equalities

$$e\cos(\omega+\pi) = -q(u) \quad \text{and} \quad e\sin(\omega+\pi) = -k(u)).$$

are true for the vector which is anticollinear to the Laplace vector. For the first of these conditions, equations (VI.1) of Appendix VI will take the form ($u_0 = 0$)

$$\left.\begin{aligned} &a_2\cos^3 u - (\gamma + a_1)\cos u + a_1 - a_2 = 0;\\ &a_2\sin^3 u + \left(\gamma - \tfrac{1}{2}\right)\sin u = 0. \end{aligned}\right\} \qquad \text{(VII.2)}$$

Here, account is taken of the fact that $b_2 = a_2$ according to (VI.2), $\varepsilon < 0$, and the notation $\gamma = e/|\varepsilon|$ is used.

In the case of the second of the conditions written above, equations (VI.1) are written in the form

$$\left.\begin{aligned} &a_2\cos^3 u - (a_1 - \gamma)\cos u + a_1 - a_2 = 0;\\ &a_2\sin^3 u - (1+\gamma)\sin u = 0. \end{aligned}\right\} \qquad \text{(VII.3)}$$

From the second equation of (VII.2), we may immediately find the one root

$$u = \pi. \qquad \text{(VII.4)}$$

The second value of the root $u = 0$ corresponds to the initial point where $e = 0$, and is taken up separately below.

The remaining two roots of the second equation in (VII.2) are determined from

$$\gamma = \frac{1}{2} - a_2 + a_2 \cos^2 u. \tag{VII.5}$$

Solving the first equation in (VII.2) and (VII.5) simultaneously, we get for the remaining two roots of system (VII.2) the expression

$$\cos u = \frac{a_2 - a_1}{a_2 - a_1 - \frac{1}{2}},$$

which cannot be used since the modulus of this fraction is greater than unity /269
at any values of i_0.

Thus, the only extremum position in this case is determined by condition (VII.4). After substitution of (VII.4) in the first equation of (VII.2), we get the expression

$$e = 2|\varepsilon|(a_2 - a_1), \tag{VII.6}$$

which may be used for determining the eccentricity of an osculating orbit at the point $u = \pi$.

This eccentricity is a monotonic function, assuming at the ends of the interval $0 \leq i \leq \pi/2$ the values $e = |\varepsilon|$ and $e = 1/2|\varepsilon|$, respectively. Consequently, the eccentricity of an osculating orbit in this range never vanishes.

Let us consider equation (VII.3) assuming condition $u_0 = 0$, and we immediately find one root of the second equation in (VII.3)

$$u = \pi$$

(the root $u = 0$, as was pointed out above, is considered separately); however, this root should be thrown out since the quantity γ must take on only negative values at $u = \pi$ according to the first equation of (VII.3). The two extremum positions in this case are determined only by the formula

$$u = \operatorname{Arccos} \frac{a_1 - a_2}{a_1 - a_2 + \frac{1}{2}}, \tag{VII.7}$$

which is obtained from the first equation in (VII.3) after eliminating the function γ by means of the equality

$$\gamma = a_2 - a_2 \cos^2 u - \frac{1}{2},$$

obtained in turn from the second equation in (VII.3). This same equality in the form

$$e = |\varepsilon| \left(a_2 \sin^2 u - \frac{1}{2} \right) \tag{VII.8}$$

determines the magnitude of the eccentricity of an osculating orbit in the apogee positions.

An investigation of formula (VII.8) assuming the conditions $|\sin u| \leq 1$, $e \geq 0$, shows that extremum positions are possible only in the range of inclinations

$$60° < i_0 < 120°. \tag{VII.9}$$

Let us now examine the positions of a satellite corresponding to the values $u = 2n\pi$[1], i.e. to the zero eccentricities of an osculating orbit of a circular satellite. Above (see Appendix VI) it was shown that at the point $u = 2\pi$ we have from the left (in the neighborhood of -2π) $\omega = 3\pi/2$, and from the right (in the neighborhood of $+2\pi$) $\omega = \pi/2$. To the left of this point, the satellite moves as if away from the perigee of the osculating orbit, i.e. $r' > 0$, while to the right it moves as if toward the perigee, and $r' < 0$. Consequently, a local maximum is reached at the points $2n\pi$, and the apogee position is passed. /270

According to (VII.4) and (VII.7), three more extremum positions will be reached in range (VII.9) on $0 \leq u \leq 2\pi$. Since these extrema must alternate,

[1] In all cases here n = 0, 1, 2,...

we conclude that the apogee position is passed at the point $u = \pi$ (just as at points $2n\pi$), and the perigee position is passed at points (VII.7).

Apart from the points $u = 2n\pi$, the eccentricity of the orbit is equal to zero at the critical inclination $i_0 = 90° \pm 19°57'$, as well as at the points $u = (2n + 1)\pi \pm 45°35'10''$. It would seem in this regard that two more additional extremum points should be passed, however, this is not so in reality. The position of the perigee points (VII.7) depends on the magnitude of the initial inclination (in terms of the coefficients a_1 and a_2), the perigee points (VII.7) converging as i_0 decreases, and moving toward the apogee point $u = \pi$. At the critical inclination, they are right on the value of the argument $u = (2n + 1)\pi \pm 45°35'10''$, which may be checked out by computation from formula (VII.7). Thus, the number of extremum points in range (VII.9) remains unchanged[1]. Further convergence (at inclinations less than critical) is accompanied by smoothing of the extrema determined by equalities (VII.4) and (VII.7) (i.e. the local maximum at $u = \pi$ becomes less pronounced). When $i_0 < 40°33'$, the extremum points disappear, and the satellite goes through only three extremum positions on the interval $0 \leq u \leq 2\pi$, the apogee point at $u = \pi$ being replaced by the perigee point. Thus, the number of extremum positions of a circular satellite on the interval $0 \leq u \leq 2\pi$ at $u_0 = 0$ is equal to 3 or 5, depending on the initial inclination.

The given analysis is illustrated by graphs in Fig. 123 produced by /271
digital computer calculations.

Let us now examine the second approach to investigation of extremum positions, based on examination of the necessary conditions $\dot{r} = 0$.

This analysis is done for satellites close to circular ($e_0 \sim 0(\varepsilon)$); the concept of circular satellites stems from this condition as a special case. Let us write the formula for the focal radius (see §12, formula (12.80)), disregarding the effect of asymmetry of the Earth relative to equatorial density,

$$r - p_0 = -e_0 p_0 \cos(\varpi + \omega_0 - u) - \frac{\varepsilon p_0}{2}\left[\frac{3}{2}\sin^2 i_0 - 1 + \left(1 - \frac{5}{4}\sin^2 i_0\right)\cos(\varpi + u_0 - u) + \frac{7}{12}\sin^2 i_0 \cos(\varpi + 3u_0 - u) + \frac{\sin^2 i_0}{6}\cos 2u - \sin^2 i_0 \cos 2u_0 + \frac{1}{2}\sin^2 u_0 \sin^2 i \sin(\varpi - u)\right], \qquad \text{(VII.10)}$$

[1] The statements relative to the number of extremum points given in [92] are /270
inaccurate. This was pointed out to the author by Yu. G. Yevtushenko.

where $\bar{\omega} = \frac{\varepsilon n_0 t}{4}(1-5\cos^2 i_0)$, $\operatorname{tg}\omega_0 = k_0/q_0$.

The first component in the right-hand member of (VII.10) appears as a consequence of orbital ellipticity. The remaining terms in (VII.10) are due to the effect of asymmetry of the gravitational field of the Earth. To determine the values of the argument of latitude corresponding to the extremum positions, let us differentiate (VII.10) with respect to u, and after setting the resultant expression equal to zero we get

$$\frac{1}{p_0}\frac{\partial r}{\partial u} = -e_0\sin(\bar{\omega}+u_0-u) - \frac{\varepsilon}{2}\left[\frac{7}{12}\sin^2 i\sin(\bar{\omega}+3u_0-u) - \right.$$
$$-\frac{\sin^2 i_0}{3}\sin 2u - \frac{1}{2}\sin^2 u_0\sin^2 i_0\cos(\bar{\omega}-u) +$$
$$\left. +\left(1-\frac{5}{4}\sin^2 i_0\right)\sin(\bar{\omega}+u_0-u)\right]+O(\varepsilon^2). \qquad \text{(VII.11)}$$

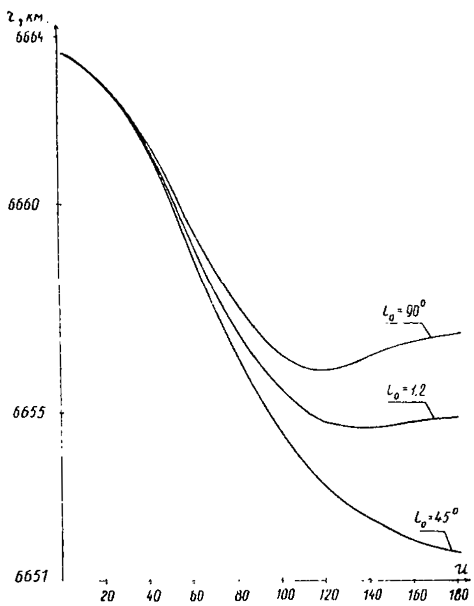


Fig. 123

As may be seen from this expression, the nature of the variation in the focal radius and the extremum positions of the satellite depend on the initial value of the argument of latitude, the angles i_0, ω_0, $\bar{\omega}$, the inclination of the orbit, the focal parameter and the initial eccentricity of the orbit. Since this relationship in the general case is complex, we shall limit ourselves as in the preceding analysis to consideration of some special cases.

We shall disregard the change in angle $\bar{\omega}$. Since $\bar{\omega}(t)$ is a slowly changing function, this simplification is permissible over a short time interval. /272

Let us assume $\omega_0 = u_0 = 0$; then, solving (VII.11), we find the four roots

$$u_1 = 0;\; u_2 = \pi;\; u_{3,4} = \operatorname{Arccos}\frac{2\varepsilon\sin^2 i_0 - 3\varepsilon - 6e_0}{2\varepsilon\sin^2 i_0}. \qquad \text{(VII.12)}$$

The last two roots take place if the inequalities

$$-1 < \frac{2\varepsilon \sin^2 i_0 - 6e_0 - 3\varepsilon}{2\varepsilon \sin^2 i_0} < 1,$$

are simultaneously fulfilled. These inequalities are equivalent to

$$2e_0 < -\varepsilon; \quad \frac{3}{4} + \frac{3e_0}{2\varepsilon} < \sin^2 i_0. \qquad \text{(VII.13)}$$

In the special case of circular satellites, we get formulas (VII.7) and (VII.9) from (VII.12) and (VII.13). If the eccentricity and orbital position are such that conditions (VII.13) are not satisfied, then three extremum positions exist: the perigee position at $u = 0, 2\pi$, and the apogee position at $u = \pi$. If conditions (VII.13) are satisfied, then two more extremum positions u_3 and u_4 appear. It may be shown that a local minimum of the focal radius is reached at these points. Thus, the satellite has perigee positions here.

The region of possible inclinations where the focal radius has four extremum positions depends on the value of the initial eccentricity. This region is a minimum when $e_0 = 0$:

$$120° > i > 60°.$$

As the eccentricity increases, this region expands. When the initial eccentricity is close to $-\varepsilon/2$, five extremum positions will take place at any inclinations not equal to zero.

Shown in Fig. 124 is the variation in the focal radius for equatorial and polar satellites at $k_0 = 0$, $p_0 = 6{,}996$ km and $e_0 = 0.0001$. In the case of an equatorial satellite, the second condition of (VII.13) is violated, and therefore, it passes through three extremum positions. In the case of a polar satellite, two more extremum positions are added to $u = 126°$ and $u = 234°$. /273

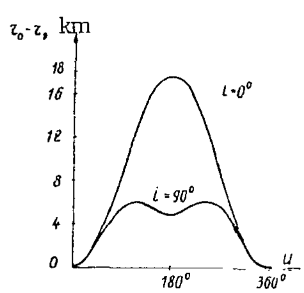


Fig. 124

A variation in the initial position of the line of aspides changes the behavior of the relationship r(u). If it is assumed that $u_0 = 0$, $\omega_0 = \pi/2$, then we find from (VII.11) that the extremum positions take place at $u = \pi/2$ and $u = 3\pi/2$, and

$$u = \operatorname{Arcsin} \frac{2\varepsilon \sin^2 i_0 - 6e_0 - 3\varepsilon}{2\varepsilon \sin^2 i_0}. \quad \text{(VII.14)}$$

And the values of the argument of latitude corresponding to the extremum positions are realized only if

$$-1 < \frac{2\varepsilon \sin^2 i_0 - 6e_0 - 3\varepsilon}{2\varepsilon \sin^2 i_0} < 1.$$

The given analysis shows that in the case of the orbits of satellites which are close to circular, just as in the case of circular orbits, there are regions in which five extremum positions are passed. In both cases, the number of extremum positions on the orbits and the values of the argument of latitude corresponding to them depend on the initial orbital parameters. /274

VIII. ANALYSIS OF THE PERTURBATIONS OF ORBITAL ELEMENTS OVER A LONG TIME INTERVAL

Let us undertake a qualitative analysis of disturbed satellite motion. We shall use the approximate solution derived in §12. Our principal attention will be devoted to studying the singularities of the evolution of Keplerian motion over a long time interval, and to an investigation of the effect of equatorial oblateness of the Earth.

Let us introduce the dimensionless quantities: $\tilde{p} = p \cdot p_0^{-1}$ --the focal parameter, $\tau = t\sqrt{\mu}\, p_0^{-3/2}$ -- the time of motion, and $\tilde{n} = (1-e^2)^{3/2}\, \tilde{p}^{-3/2}$ --the mean angular velocity.

We shall conduct our investigation over the long time interval $\tau \sim \varkappa^{-1}\tilde{n}^{-1}$ (where $\kappa = -\varepsilon$), where the satellite makes $\sim\kappa^{-1}\tilde{n}^{-1}$ revolutions about the Earth, i.e. for nearly circular orbits of the order of 100 revolutions close to the Earth, and for 24-hour orbits of the order of 3,000 revolutions.

Let us write the approximate formulas for calculating motion of the satellite

$$\left.\begin{aligned} &\Omega = \Omega_0 - \frac{\tau\varkappa}{2}\tilde{n}_0 \cos i_0, \quad i = i_0, \quad \tilde{p} = 1; \\ &q = e_0 \cos\omega, \quad k = e_0 \sin\omega; \\ &L = L_0 + \tau\left[\tilde{n}_0 + \theta + \frac{\varkappa}{2}\tilde{n}_0^{1/3} R_0^3 (1 - 3\sin^2 u_0 \sin^2 i_0)\right], \end{aligned}\right\} \quad \text{(VIII.1)}$$

where $\omega = \theta\tau + \omega_0$, $\theta = \frac{\tilde{n}_0 \varkappa}{4}(5\cos^2 i_0 - 1)$, $\operatorname{tg}\omega_0 = \frac{k_0}{q_0}$, $R_0 = 1 + q_0 \cos u_0 + k_0 \sin u_0$.

The zero subscript indicates the initial values of the functions.

The angle L is related to the argument of latitude and the true anomaly M by the expressions

$$L = 2\operatorname{arctg}\sqrt{\frac{1-e}{1+e}}\frac{q\sin u - k\cos u}{e + q\cos u + k\sin u} - \frac{\sqrt{1-e^2}(q\sin u - k\cos u)}{1 + q\cos u + k\sin u} + \omega; \quad \text{(VIII.2)}$$

$$L = M + \omega. \quad \text{(VIII.3)}$$

Solution (VIII.1) approximates the exact solution of the problem of satellite motion with error $0(\kappa\tilde{n}^{-1})$. In the following discussion, we shall call (VIII.1) a first-approximation solution. The approximate solution describes the average effect of disturbances for a large number of satellite revolutions. The more exact solution, which reflects the singularities of disturbances over the extent of each revolution taken individually, gives a /275 second-approximation solution.

It follows from (VIII.1) that in the first approximation (i.e. with error $0(\kappa)$) the orbital eccentricity, focal parameter and inclination remain constant. Flattening of the Earth causes rotation of the line of aspides in the plane of the orbit with a constant angular velocity θ. At inclination equal to $i = i_0^* = 63.4°$, the line of apsides does not change its angular position with respect to the line of nodes. If the orbital inclination is less than i_0^*, then $\theta > 0$ and the angular distance of the perigee from the line of nodes increases. At inclinations greater than i_0^*, the angular distance ω decreases.

For near-circular satellites whose initial eccentricity has the order of oblateness, the components q, k of the Laplace vector, according to (VIII.1), will remain quantities of the first negative order of magnitude throughout the entire interval of motion, and therefore, they may be considered constant in the first approximation. In this case, we cannot discuss the behavior of the line of apsides on the basis of formulas (VIII.1),. The nature of the variation in the angular position of the perigee may be studied only with the aid of second-approximation formulas which account for short-period disturbances in the orbital elements.

It is apparent from the first equation of (VIII.1) that the line of apsides rotates under the effect of flattening of the Earth at constant angular velocity with respect to the polar axis of the Earth. The speed of rotation of the line of nodes is considerably dependent on orbital inclination (we have already pointed this out in §8). For equatorial orbits (i = 0), the line of nodes rotates at a maximum angular velocity equal to $\kappa\tilde{n}_0/2$, while for polar satellites, the line of nodes is stationary.

The angular rotational velocity of the mean anomaly in disturbed motion decreases or increases as a function of the sign of the difference $1 - 3 \sin^2 u_0 \sin^2 i_0$.

For satellites close to the Earth (with orbital radii of 6,500 - 7,000 km), the rate of change in the quantities Ω, ω, M may reach 5° per day as a consequence of flattening. An increase in the orbital radius causes a reduction in the disturbing effects.

Let us find an expression for the draconic (nodal) period. Let us assume that the satellite is on the line of nodes ($u = 0$) at some time τ_0. If T_1 is the draconic period, then the satellite again intersects the line of nodes at $\tau_0 + T_1$ ($u = 360°$). From the last formula of (VIII.1), we find the expression for the change in the angle L in time T_1:

$$L(\tau_0+T_1)-L(\tau_0)=T_1\left[\tilde{n}_0+\theta+\frac{\kappa\tilde{n}_0^{1/3}}{2}R_0^3(1-3\sin^2 u_0\sin^2 i_0)\right]. \tag{VIII.4}$$ /276

Let us transform (VIII.2) to the form

$$L=2\,\mathrm{arctg}\sqrt{\frac{1-e}{1+e}}\,\mathrm{tg}\,\frac{u-\omega}{2}-\frac{e\sqrt{1-e^2}\sin(u-\omega)}{1+e\cos(u-\omega)}+\omega. \tag{VIII.5}$$

According to (VIII.1), the change in angle ω over period T_1 will be a small quantity of order κ. Therefore, expanding (VIII.5) in a series with respect to κ we get

$$L(\tau_0+T_1)-L(\tau_0)=2\pi+\frac{\partial L}{\partial\omega}\Big/_{u=0}\theta T_1+0(\kappa^2). \tag{VIII.6}$$

Differentiating (VIII.5) with respect to ω, we find

$$\frac{\partial L}{\partial\omega}\Big/_{u=0}=1-\frac{(1-e^2)^{3/2}}{(1+e\cos\omega)^2}. \tag{VIII.7}$$

Let us substitute (VIII.7) in (VIII.6). Then equating the right-hand members of (VIII.4) and (VIII.6), we get an expression for the draconic period

$$T_1 = \frac{2\pi}{\tilde{n}_0}\left\{1-\frac{\varkappa}{2}\left[\frac{R_0^3(1-3\sin^2 u_0 \sin^2 i_0)}{1-e_0^2}+\frac{(1-e^2)^{3/2}(5\cos^2 i_0-1)}{2(1+e_0\cos\omega)^2}\right]\right\}. \qquad \text{(VIII.8)}$$

For orbits with low eccentricity, $e \sim 0(\kappa)$ we get from (VIII.8) a formula which was found previously in [89]:

$$T_1 = T_0\left\{1 - \frac{\varkappa}{4}[1+5\cos^2 i_0 - 6\sin^2 u_0 \sin^2 i_0]\right\}, \qquad \text{(VIII.9)}$$

where $T_0 = 2\pi/\tilde{n}_0$ is the period of Keplerian motion.

The length of the draconic period depends on orbital inclination, the initial value of u_0 and the position of the line of apsides. The initial value of the argument of latitude has a considerable effect on the length of the draconic period, particularly at large orbital inclinations.

Shown in Fig. 125 is the difference between the initial (undisturbed) period of orbital motion and the disturbed draconic period for a circular satellite with a radius equal to the equatorial radius of the Earth as a function of i_0.

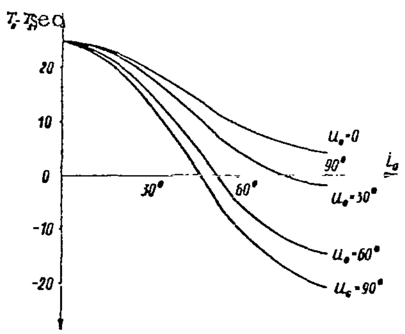


Fig. 125

For polar orbits, the draconic period may change by 25 seconds, depending on u_0. An examination of Fig. 125 shows that a combination of values i_0, u_0 exists /277 such that for this combination the period T_0 is equal to the disturbed draconic period T. For near-circular satellites, this takes place when

$$\sin^2 u_0 = -\frac{5}{6}+\frac{1}{\sin^2 i_0}. \qquad \text{(VIII.10)}$$

Equality (VIII.10) becomes significant when

$$\sin^2 i_0 \geq \frac{6}{11}. \qquad \text{(VIII.11)}$$

The regions of values of i_0 as a function of u_0 where the draconic period T_1 is less than, equal to or greater than the undisturbed period T_0 for near-circular orbits are shown in Fig. 126.

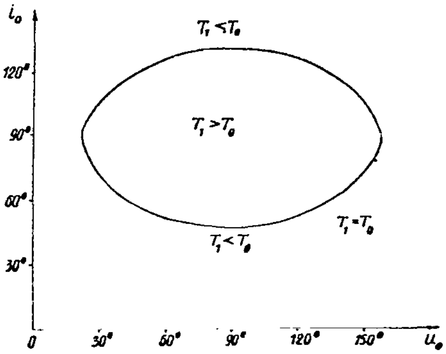


Fig. 126

It follows from formula (VIII.8) that if the satellite was on the line of nodes at the initial instant, then the undisturbed period is greater than the draconic period when

$$\frac{2R_0^3(1+e\cos\omega)^2}{(1-e_0^2)^{3/2}} > 1-5\cos^2 i_0. \qquad \text{(VIII.12)}$$

For inclinations smaller than $i_0^* = 63.4°$, the right-hand member of (VIII.12) is negative, and therefore, the inequality takes place at any eccentricities. Thus, if $u_0 = 0$ and $i_0 < i_0^*$, then the draconic period is less than the Keplerian period. However, for orbits close to polar, the draconic period may be greater than the Keplerian period. In particular, if the secular change in ω is disregarded (which is a permissible simplification over long time periods), and it is assumed that $\omega = \omega_0 = \pi$, then we find from (VIII.8) that the draconic period is greater than the Keplerian period for polar orbits when

$$\frac{R_0^3}{1-e_0^2} < \frac{(1-e^2)^{3/2}}{2(1-e)^2}. \qquad \text{(VIII.13)}$$

Let us solve (VIII.13). Assuming that $R_0 = 1 - e_0$, we get

$$e > 0{,}242. \qquad \text{(VIII.14)}$$

In the course of time, the line of apsides changes its position, which leads to a change in condition (VIII.14). In particular, for the case $\omega_0 = \pi$, $\omega = 0$, $i = 90°$, the draconic period is greater than the Keplerian if $e > 0.6$.

Thus, over a long interval of motion, it may turn out that the Keplerian period of motion was originally less than the draconic, and then due to evolution of the orbit, the draconic period becomes greater than the initial /278 undisturbed period. If the period of motion of the satellite is equal to or close to 12 or 24 hours, then resonance effects appear due to the ellipticity of the equator. In the first approximation, the resonance effects show up in a change of solution $L(\tau)$.

Let us examine the motion of a satellite around the Earth with a period close to 24 hours in an orbit of low eccentricity $e^3 \sim 0(\kappa)$.

Let us use the symbol α to denote the angle between the mean longitude of the satellite, equal to $L + \Omega$, and the longitude of the semiminor axis of the equatorial ellipse:

$$\alpha = \frac{\pi}{2} + L - \Omega - \bar{\omega}_3 \tau,$$

where $\bar{\omega}_3$ is the dimensionless angular rotational velocity of the Earth around the polar axis, related to the dimensional angular velocity by the formula

$$\bar{\omega}_3 = \omega_3 \mu^{-1/2} p_0^{3/2} .$$

The angle α satsifies the equation (see §12)

$$\frac{d^2\alpha}{d\tau^2} = -\frac{3\kappa^2 b}{8}\left(1 - \frac{15}{2} e^2\right)[9e^2 \sin^2 i \sin(2\alpha - 2\omega) + 2(1 + \cos i)^2 \sin 2\alpha]. \tag{VIII.15}$$

Equation (12.28) describes appreciably nonlinear oscillations or rotations of the mean longitude of the satellite relative to the semiminor axis of the equatorial ellipse. Low-amplitude resonance oscillations were studied in [90] and [91] for the case of near-circular orbits with low inclination. Oscillations of an equatorial satellite were studied in the nonlinear formulation by Perkins [92]. The equations obtained in these works for the longitude of the satellite are special cases of equation (VIII.11).

For orbits of low eccentricity, when $e^2 \sim O(\kappa)$, equation (12.28) is simplified:

$$\frac{d^2\alpha}{d\tau^2} = -\frac{3\varkappa^2 b(1+\cos i_0)^2}{4}\sin 2\alpha . \qquad \text{(VIII.16)}$$

Let us use the symbol κF to denote the complete energy of the satellite:

$$2\varkappa F = \tilde{n}^{2/3} - \tilde{n}_0^{2/3} + 2\varkappa(U - U_0),$$ /279

where

$$U = \frac{1}{6\tilde{r}^3}(1-3\sin^2\varphi) + \frac{\varkappa c}{20\tilde{r}^5}[35\sin^4\varphi - 30\sin^2\varphi + 3] + $$
$$+ \frac{\varkappa f}{2\tilde{r}^4}(5\sin^2\varphi - 3)\sin\varphi + \frac{\varkappa b}{2\tilde{r}^3}\cos 2\Lambda\cos^2\varphi;$$

$$\tilde{r} = r\cdot p_0^{-1};\quad c = \frac{5c_{40}r_0^4}{2\varkappa^2 p_0^4};\quad c_{30} = \frac{f\varkappa^2 p_0^3}{r_0^3};\quad b = \frac{6r_0^2}{\varkappa^2 p_0^2}\sqrt{c_{22}^2 + d_{22}^2}.$$

Here, Δ is the longitude of the satellite reckoned from the major axis of the equatorial ellipse. In the nonresonance case (when the period of orbital motion is not equal to 12 or 24 hours), satellite motion takes place with constant total energy with an accuracy to quantities of order $\sim\kappa$. In the resonance case, F satisfies the equation

$$\frac{dF}{d\tau} = -\frac{\varkappa b\bar{\omega}_3}{8}\left(1 - \frac{11e^2}{2}\right)[9e^2\sin^2 i\sin(2\alpha + 2\omega) + 2(1+\cos i)^2\sin 2\alpha].$$

Setting the right-hand members of (VIII.16) equal to zero (for the case of near-circular orbits $e \sim \kappa$), we find that motion with constant energy (stationary resonance conditions) takes place only when $\alpha = 0, \pi/2, \pi, 3\pi/2$, i.e. only in the case where the mean longitude of the satellite coincides with the longitude of the semiminor or semimajor axes of the equatorial ellipse. It follows from (VIII.15) that for large eccentricities ($e^3 \sim O(\kappa)$), stationary conditions exist only for orbits with an inclination close to 63.4° (since only in this case is there no secular variation in the angle ω, and the equation (VIII.15) has zero solutions).

Let us investigate the stability of stationary resonance conditions. We shall use the notation α_* to denote any of the values of α which correspond to stationary resonance conditions. Setting up an equation in variations for (VIII.15) and (VIII.16), we get

$$\frac{d^2\delta\alpha}{d\tau^2}+\frac{3\varkappa^2\dot{b}}{4}\left(1-\frac{15}{2}e^2\right)[9e^2\sin^2 i\cos(2\alpha_*-2\omega)+2(1+\cos i)^2\cos 2\alpha_*]\delta\alpha=0;$$
$$\frac{d^2\delta\alpha}{d\tau^2}+\frac{3\varkappa^2 b(1+\cos i)^2}{2}\,\delta\alpha\cdot\cos 2\alpha_*=0.$$

The characteristic equations for (VIII.15) and (VIII.16) will be

$$\lambda^2+\frac{3\varkappa^2 b}{4}\left(1-\frac{15}{2}e^2\right)[9e^2\sin^2 i\cos(2\alpha_*-2\omega)+2(1+\cos i)^2\cos 2\alpha_*]=0; \qquad \text{(VIII.17)}$$
$$\lambda^2+\frac{3\varkappa^2 b(1+\cos i_0)^2}{2}\cos 2\alpha_*=0. \qquad \text{(VIII.18)}$$

If equations (VIII.17), (VIII.18) have real roots, then conditions α_* are /280 unstable; if the roots of (VIII.17) and (VIII.18) are imaginary, then the stationary conditions will be stable. Corresponding to stable conditions on the phase plane (α, $\dot{\alpha}$) are singular points of the focus type, unstable saddle points. In the case of unstable conditions, the roots of (VIII.17) and (VIII.18) are real and different, and therefore the angle α close to the saddle point increases exponentially with a growth index proportional to κ. In the stable case, the angle α oscillates with respect to $\alpha = 0$, π with a low frequency proportional to κ.

Thus, the satellite moves with a constant mean longitude relative to the rotating Earth only in the case of stationary resonance orbits when its mean longitude coincides with the directions of the equatorial axes. Orbits in which the mean longitude coincides with that for the minor axis are stable, while if the mean longitude coincides with that of the major equatorial axis, then these orbits are unstable.

From the condition $\dot{\alpha} = 0$ (see 12.35), we find the value of the initial mean angular velocity of motion of the satellite $\tilde{n}_0$ where stationary conditions are possible

$$\tilde{n}_0=\overline{\omega}_3+\frac{\varkappa\overline{\omega}_3}{4}(1-3\cos^2 i_0)-\frac{\varkappa R_0^3\overline{\omega}_3^{1/3}}{2}(1-3\sin^2 i_0\sin^2 u_0)+O(\varkappa^2).$$

Let us find the analytical solution for equation (VIII.12). Multiplying both members of (VIII.12) by $d\alpha/d\tau$ and integrating, we get

$$\frac{1}{2}\left(\frac{d\alpha}{d\tau}\right)^2 = \frac{3\varkappa^2 b}{8}(1+\cos i_0)^2\cos 2\alpha + c_1. \qquad \text{(VIII.19)}$$

Let $\dot{\alpha}_0$ designate the initial angular velocity of the angle α at the instant when the satellite is on the minor axis of the equatorial ellipse. Assuming

$$\tilde{k}^2 = \frac{2\dot{\alpha}_0^2}{3\varkappa^2 b(1+\cos i_0)^2}, \qquad \text{(VIII.20)}$$

we rewrite (VIII.19) in the following form:

$$\frac{1}{2}\left(\frac{d\alpha}{d\tau}\right)^2 = \frac{3\varkappa^2 b}{4}(1+\cos i_0)^2(\tilde{k}^2 - \sin^2\alpha). \qquad \text{(VIII.21)}$$

Three types of motion are possible depending on the quantity $\tilde{k}$: $\tilde{k} < 1$, $\tilde{k} > 1$ and $\tilde{k} = 1$.

/281

1. $\tilde{k} < 1$. In this case, we set

$$\tilde{k} = \sin\alpha_1. \qquad \text{(VIII.22)}$$

It follows from (VIII.21) that at the instant when α reaches α_1, the angular velocity $\dot{\alpha}$ becomes zero, and the direction of motion of the angle α reverses. The angle α will oscillate between α_1 and $-\alpha_1$. Integrating (VIII.21), we find

$$\tau = \frac{2}{\varkappa\sqrt{6b}(1+\cos i_0)}\int_0^\alpha \frac{d\alpha}{\sqrt{\sin^2\alpha_1 - \sin^2\alpha}}. \qquad \text{(VIII.23)}$$

Integral (VIII.23) is a complete elliptical integral of the first kind. By means of the transformation $\sin\alpha_1 \cdot \sin\phi = \sin\alpha$, we reduce integral (VIII.23) to the normal form:

$$\tau = \frac{2}{\varkappa(1+\cos i_0)\sqrt{6b}}\int_0^\psi \frac{d\varphi}{\sqrt{1-\tilde{k}^2\sin^2\varphi}} = \frac{2}{\varkappa(1+\cos i_0)\sqrt{6b}}F(\varphi,\tilde{k}). \qquad \text{(VIII.24)}$$

Inversion of this relationship by means of the Jacobian elliptic function $\mathrm{sn}(\tilde{k},\tau)$ gives first

$$\sin\varphi = \mathrm{sn}\left(\tilde{k}, \frac{\tau\varkappa}{2}\sqrt{6b}\,(1+\cos i_0)\right),$$

from which it then follows that

$$\alpha = \arcsin\left[\sin\alpha_1 \cdot \mathrm{sn}\left(\tilde{k}, \frac{\tau\varkappa}{2}\sqrt{6b}\,(1+\cos i_0)\right)\right]. \qquad \text{(VIII.25)}$$

The period of oscillations of the angle α is determined by the equality

$$T = \frac{8K(\tilde{k})}{\varkappa(1+\cos i_0)\sqrt{6b}}, \qquad \text{(VIII.26)}$$

where $K(\tilde{k})$ is a complete elliptic integral of the first kind. If the initial angular velocity $\dot{\alpha}_0$ is small, then according to (VIII.20), the modulus of the elliptic integral is small. In this case, the satellite makes small oscillations (librations) relative to the minor axis of the equatorial ellipse. Let us use the known expansion of the complete elliptic integral with respect to the modulus. Then, from (VIII.26) we get the following formula for the period of oscillations with low amplitude:

$$T = \frac{4\pi}{\varkappa(1+\cos i_0)\sqrt{6b}}\left(1+\frac{\alpha_1^2}{4}\right). \qquad \text{(VIII.27)}$$

The period of the librations of angle α is inversely proportional to χ, and therefore, is a large quantity. Consequently, the ellipticity of the equator results in a long-period oscillation of the mean longitude of the satellite /282
relative to the minor equatorial axis. Returning to dimensional quantities, we get from (VIII.26) the following formula for the period of the oscillations with respect to the variable t:

$$T = \frac{4Kp_0^{5/2}}{3r_0\sqrt{\mu}\,\sqrt[4]{c_{22}^2+d_{22}^2}}.$$

We compute the period of small oscillations from the formula

$$T = \frac{2\pi p_0^{5/2}}{3r_0\sqrt{\mu}\sqrt[4]{c_{22}^2+d_{22}^2}}. \tag{VIII.28}$$

If we assume $6\sqrt{c_{22}^2+d_{22}^2} = 10^{-5}$, then it follows from (VIII.28) that the period is equal to 855 days, or 2.3 years. The period increases with inclination, and is twice as long for a polar orbit as for an equatorial orbit. Shown in Fig. 127 is the relationship between T and the maximum amplitude of oscillations in α_1 for equatorial and polar satellites. The period of the librations tends to ∞ as α_1 approaches $\pi/2$.

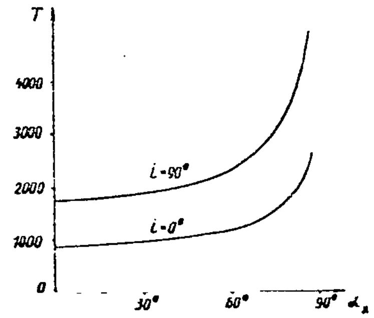


Fig. 127

2. $\tilde{k} > 1$. In this case, setting

$$k_1^2 = \frac{3\varkappa^2 b(1+\cos i_0)^2}{2\dot{\alpha}_0^2}, \tag{VIII.29}$$

we represent equation (VIII.21) in the form

$$\frac{1}{4}\left(\frac{d\alpha}{d\tau}\right)^2 = \frac{\dot{\alpha}_0^2}{4}[1-k_1^2\sin^2\alpha]. \tag{VIII.30}$$

The quantity k_1 is less than unity, and consequently, the angular velocity $\dot{\alpha}$ is nowhere equal to zero. Therefore, instead of oscillating, the angle α will rotate in one and the same direction.

Integrating (VIII.30), we get

$$\tau = \frac{1}{\dot{\alpha}_0}\int_0^{\alpha}\frac{d\alpha}{\sqrt{1-k_1^2\sin^2\alpha}}, \tag{VIII.31}$$

Inverting integral (VIII.31), we find

$$\alpha = \mathrm{am}\,\dot{\alpha}_0\tau. \tag{VIII.32}$$ /283

The period of rotation is determined by the equality

$$T = \frac{2}{\dot{\alpha}_0} \int_0^{\pi/2} \frac{d\alpha}{\sqrt{1-k_1^2 \sin^2 \alpha}} = \frac{2K(k_1)}{\dot{\alpha}_0}. \tag{VIII.33}$$

For large values of $\dot{\alpha}_0$, (VIII.33) may be expanded in a series with respect to k_1.

Dropping terms of order k_1^4, we get

$$T = \frac{\pi}{\dot{\alpha}}\left(1 + \frac{k_1^2}{4}\right). \tag{VIII.34}$$

If $\dot{\alpha}_0 > 0$, then the angle α increases monotonically, and the satellite moves in the westward direction relative to the equatorial ellipse. If $\dot{\alpha}_0 < 0$, then motion is constant with respect to direction. The semimajor axis of the satellite's orbit is inversely proportional to its angular velocity, and therefore, corresponding to the case $\dot{\alpha}_0 > 0$ are orbits with a shorter focal radius than in the case $\dot{\alpha}_0 < 0$.

3. $\tilde{k} = 1$. Equation (VIII.21) in this case assumes the form /284

$$\frac{1}{2}\left(\frac{d\alpha}{d\tau}\right) = \frac{3\varkappa^2 b (1+\cos i_0)^2}{4} \cos^2 \alpha. \tag{VIII.34'}$$

Integrating (VIII.34'), we get

$$\tau = \frac{2}{\varkappa(1+\cos i_0)\sqrt{6b}} \ln\left[\operatorname{tg}\left(\frac{\alpha}{2} + \frac{\pi}{4}\right)\right]. \tag{VIII.35}$$

This formula shows that as τ increases from 0 to ∞, the angle α will increase monotonically from 0 to π. Thus, if the mean longitude of the satellite conicided with that of the semiminor axis of the equatorial ellipse at the initial instant of motion, then the reverse direction of the semiminor axis will be the limiting position of α.

Shown in Fig. 128 is part of the phase plane for an equatorial satellite. The angle α and angular velocity $\dot{\alpha}$ are taken as the axes. The phase trajectories $\alpha(t)$, $\dot{\alpha}(t)$ are periodic with respect to α with period $\pi/2$. The arrow indicates the direction of motion. The broken line is used as a separatrix. The phase trajectories lying beneath the separatrix correspond to the first case--oscillations about the stable position $\alpha = 0$. The trajectories lying above the separatrix are the second case--rotational motions.

The separatrix corresponds to the third case of aperiodic motions.

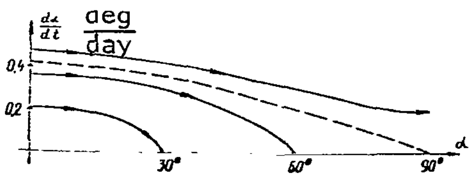


Fig. 128

The law of variation in the total energy of the satellite may be found from (12.14). In particular for near-circular orbits we get

$$F = \frac{1}{3\varkappa \tilde{n}_0^{1/3}} \Big[\frac{d\alpha}{d\tau} + \bar{\omega}_3 - \tilde{n}_0 - \frac{\varkappa^2 b \bar{\omega}_3 \cos 2\alpha}{2} (1+\cos i_0)(1+2\cos i_0) - $$

$$ - \frac{\varkappa \tilde{n}_0}{4}(1-5\cos^2 i_0) - \frac{\varkappa \tilde{n}_0^{1/3}}{2}(1-3\sin^2 u_0 \sin^2 i_0) \Big] . \quad \text{(VIII.36)}$$

The solutions of (VIII.25) and (VIII.32) should be substituted in the right-hand member of (VIII.36). It follows from (VIII.36) that the total energy in the resonance case is a slowly varying periodic function of time.

Appendices to Chapter Three /285

IX. EQUATIONS OF MOTION OF A CONTROLLED SPACE VEHICLE IN THE FIELD OF THE SPHEROIDAL EARTH

In studying the motion of a controlled space vehicle close to the Earth, the required degree of precision will be achieved when the gravitational potential is written in the form of Model Б. The equations of motion in osculating elements with respect to the argument t in this case take the form

$$\left.\begin{aligned}
&\frac{d\Omega}{dt}=\sqrt{p\mu}R^{-1}\frac{\sin u}{\sin i}W;\\
&\frac{di}{dt}=\sqrt{p\mu}R^{-1}\cos uW;\\
&\frac{dp}{dt}=2\sqrt{p\mu}R^{-1}pT;\\
&\frac{dq}{dt}=\sqrt{p\mu}\{kR^{-1}\operatorname{ctg} i\sin uW+[(q+\cos u)R^{-1}+\cos u]T-\sin uS\};\\
&\frac{dk}{dt}=\sqrt{p\mu}\{-q\operatorname{ctg} iR^{-1}\sin uW+[(k+\sin u)R^{-1}+\sin u]T+\cos uS\};\\
&\frac{du}{dt}=\sqrt{p\mu}(p^{-2}R^2-R^{-1}\operatorname{ctg} i\sin uW).
\end{aligned}\right\} \qquad \text{(IX.1)}$$

Here $R = 1 + q\cos u + k\sin u$;

$$\varepsilon = 3c_{20}r_0^2/p_0^2;\ S=\tfrac{1}{2}\varepsilon p^{-2}R^4(3\sin^2 i\sin^2 u-1)-S^*;$$
$$T=\tfrac{1}{2}\varepsilon p^{-2}R^4\sin i\sin 2u+T^*;\ W=\tfrac{1}{2}\varepsilon p^{-2}R^4\sin 2i\sin u+W^*;$$

S^*, T^* и W^*

are, respectively equal to

$$S^* = a_r/\mu;\quad T^* = a_\tau/\mu;\quad W^* = a_n/\mu,$$

where a_r, a_τ and a_n are the projections of the controlling or correcting acceleration on the focal radius, the transversal and the positive normal to the plane of motion. /286 System (IX.1) may be used for describing the motion of an artificial Earth satellite with regard to other perturbing effects than the first-order polar oblateness of the Earth. Then, the components of the corresponding effects are written as S*, T* and W*.

This same system of equations may be written with respect to the argument of latitude u:

$$
\left.\begin{aligned}
&\frac{d\Omega}{du}=\varepsilon R\cos i\sin^2 u+\frac{p^2}{\sin i}R^{-3}\sin u W^*;\\
&\frac{di}{du}=\frac{1}{4}\varepsilon R\sin 2i\sin 2u+p^2R^{-3}\cos u W^*;\\
&\frac{dp}{du}=\varepsilon R\sin^2 i\sin 2u+2p^3R^{-3}T^*;\\
&\frac{dq}{du}=\frac{1}{2}\varepsilon R\{2k\cos^2 i\sin^2 u+(q+\cos u+R\cos u)\sin i\sin 2u-\\
&\quad -R(3\sin^2 i\sin^2 u-1)\sin u\}+\sqrt{p\mu}\{k\sin u\operatorname{ctg} iR^{-1}W^*+\\
&\quad +[R^{-1}(q+\cos u)+\cos u]T^*+S^*\sin u\};\\
&\frac{dk}{du}=\frac{1}{2}\varepsilon R\{-2q\cos^2 i\sin^2 u+(k+\sin u+R\sin u)\sin i\sin 2u+\\
&\quad +R(3\sin^2 i\sin^2 u-1)\cos u\}+\sqrt{p\mu}\{-q\sin u\operatorname{ctg} iR^{-1}W^*+\\
&\quad +[(k+\sin u)R^{-1}+\sin u]T^*-S^*\cos u\};\\
&\frac{dt}{du}=\frac{p}{R^2}\sqrt{\frac{p}{\mu}}[1+\varepsilon R\cos^2 i\sin^2 u+p^2R^{-3}\operatorname{ctg} i\sin u W^*].
\end{aligned}\right\}\qquad\text{(IX.2)}
$$

X. ON THE METHOD OF AVERAGING SYSTEMS WITH A RAPIDLY ROTATING PHASE

The asymptotic methods of the theory of differential equations (methods of averaging) are an effective tool for solving a number of problems in nonlinear mechanics. The method of averaging consists of using special substitution of variables to reduce complex systems of differential equations to simpler averaged systems. The method of averaging has been strictly formulated and substantiated by N. M. Krylov and N. N. Bogolyubov. The method has been further developed and generalized in works by Yu. A. Mitropol'skiy, V. M. Volosov and a number of other authors [64 - 67].

Let us briefly outline V. M. Volosov's method of averaging, which is a generalization of the procedure given by N. N. Bogolyubov and D. N. Zubarev [65] for averaging systems with a rapidly rotating phase. /287

Let there be a system of the form

$$\dot{x}=\varepsilon X_1(x,y)+\varepsilon^2X_2(x,y)+\varepsilon^3\ldots;\qquad\text{(X.1)}$$

$$\dot{y} = n + \varepsilon Y_1(x, y) + \varepsilon^2 Y_2(x, y) + \varepsilon^3 \dots , \tag{X.2}$$

where ε is a small parameter ($\varepsilon \ll 1$); x, X_1, X_2,... are m-dimensional vector functions $x = \{x_1, \dots, x_m\}$, $X_1 = \{X_{11}, \dots, X_{1m}\}$, $X_2 = \{X_{21}, \dots, X_{2m}\} \dots$; y, Y_1, Y_2,... are k-dimensional vector functions. To simplify the computations, we shall set $k = 2$. We shall assume that the vector $n = \{n_1, n_2\}$ is constant. In [67], the more general case is considered where n depends on x, y, t. The functions X_1, X_2, Y_1, Y_2,... are periodic with respect to y with periods 2π. The right-hand members of the equations of system (X.1) are proportional to the small parameter ε, and therefore, variables x are slowly changing functions. The variables[1] change comparatively more rapidly since $\dot{y} \sim n \gg \varepsilon$.

We shall attempt with the aid of special substitution of variables to reduce system (X.1), (X.2) to a simpler system in which the rapidly and slowly changing variables would be separated. We shall seek the substitution of variables in the form

$$\left.\begin{aligned} x &= \bar{x} + \varepsilon u_1(\bar{x}, \bar{y}) + \varepsilon^2 u_2(\bar{x}, \bar{y}) + \dots + \varepsilon^l u_l(\bar{x}, \bar{y}) + \dots ; \\ y &= \bar{y} + \varepsilon v_1(\bar{x}, \bar{y}) + \varepsilon^2 v_2(\bar{x}, \bar{y}) + \dots + \varepsilon^l v_l(\bar{x}, \bar{y}) + \dots , \end{aligned}\right\} \tag{X.3}$$

where u_1, u_2,... are m-dimensional and v_1, v_2,... are k-dimensional as yet undefined vector functions. We shall consider functions u_1, u_2, v_1, v_2,... periodic with respect to y with periods 2π. After the substitution of variables (X.3) the system is reduced to the averaged form

$$\left.\begin{aligned} \dot{\bar{x}} &= \varepsilon A_1(\bar{x}) + \varepsilon^2 A_2(\bar{x}) + \varepsilon^3 \dots ; \\ \dot{\bar{y}} &= n + \varepsilon B_1(\bar{x}) + \varepsilon^2 B_2(\bar{x}) + \varepsilon^3 \dots . \end{aligned}\right\} \tag{X.4}$$

The functions A_1, A_2, B_1, B_2 will be defined below. System (X.4) is considerably simpler than the initial system (X.1), (X.2), since the equations for rapidly changing variables $\bar{x}$ and slowly changing variables $\bar{y}$ are separated in (X.4). Therefore, the equations with respect to $\bar{x}$ may be integrated independently of the last equations in this system. After this, finding the solution of $\bar{y}(t)$ reduces to computing a quadrature. Substitution (X.3) may be interpreted as expansion of the real solution of system (X.1), (X.2) into /288

[1]This must be a misprint and should read: "... variables y change...".

the averaged solution $\bar{x}$, $\bar{y}$ and the short-period solution described by $u_1(y)$, $u_2(y)$, $v_1(y)$,... Differentiating (X.3) on the strength of (X.4), we get

$$\dot{\bar{x}} = \varepsilon A_1(\bar{x}) + \varepsilon \frac{\partial u_1}{\partial \bar{y}} n + \varepsilon^2 \frac{\partial u_1}{\partial \bar{y}} B_1 + \varepsilon \frac{\partial u_1}{\partial \bar{x}} A_1 + \varepsilon^2 \frac{\partial u_1}{\partial \bar{x}} A_2 + \dots . \tag{X.5}$$

Here $\partial/\partial\bar{x}$ denotes the differential operator: $\dfrac{\partial}{\partial \bar{x}} = \left\{\dfrac{\partial}{\partial \bar{x}_1}, \dfrac{\partial}{\partial \bar{x}_2}, \dots, \dfrac{\partial}{\partial \bar{x}_m}\right\}$.

Therefore expression $\dfrac{\partial u_1}{\partial \bar{x}} A_1$ when written out in more detail has the form

$$\frac{\partial u_1}{\partial \bar{x}} A_1 = \frac{\partial u_1}{\partial \bar{x}_1} A_{11} + \frac{\partial u_1}{\partial \bar{x}_2} A_{12} + \dots + \frac{\partial u_1}{\partial \bar{x}_m} A_{1m}.$$

The remaining expressions in (X.5) may be similarly expanded. Let us substitute (X.5) in (X.1), (X.2) and expand the functions $X_1(\bar{x}, \bar{y})$, $X_2(\bar{x}, \bar{y})$, $Y_1(\bar{x}, \bar{y})$,... in a series in powers of ε. Then equating the coefficients associated with identical powers of ε, we get the infinite system of equations

$$\left.\begin{aligned}
&A_1(\bar{x}) + \frac{\partial u_1(\bar{x},\bar{y})}{\partial \bar{y}} n = X_1(\bar{x},\bar{y}); \quad B_1(\bar{x}) + \frac{\partial v_1(\bar{x},\bar{y})}{\partial \bar{y}} n = Y_1(\bar{x},\bar{y});\\
&A_2(\bar{x}) + \frac{\partial u_1(\bar{x},\bar{y})}{\partial \bar{y}} B_1(\bar{x}) + \frac{\partial u_1(\bar{x},\bar{y})}{\partial \bar{x}} A_1(\bar{x}) + \frac{\partial u_2}{\partial \bar{y}} n =\\
&= \frac{\partial X_1(\bar{x},\bar{y})}{\partial \bar{x}} u_1(\bar{x},\bar{y}) + \frac{\partial X_1(\bar{x},\bar{y})}{\partial \bar{y}} v_1(\bar{x},\bar{y}) + X_2(\bar{x},\bar{y});\\
&B_2(\bar{x}) + \frac{\partial v_1(\bar{x},\bar{y})}{\partial \bar{y}} B_1(\bar{x}) + \frac{\partial v_1(\bar{x},\bar{y})}{\partial \bar{x}} A_1(\bar{x}) + \frac{\partial v_2(\bar{x},\bar{y})}{\partial \bar{y}} n =\\
&= \frac{\partial Y_1(\bar{x},\bar{y})}{\partial \bar{x}} u_1(\bar{x},\bar{y}) + \frac{\partial Y_1(\bar{x},\bar{y})}{\partial \bar{y}} v_1(\bar{x},\bar{y}) + Y_2(\bar{x},\bar{y});\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots
\end{aligned}\right\} \tag{X.6}$$

Solving system (X.6) we sequentially determine the functions A_1, B_1, u_1, v_1, u_2,..., after which the problem of integrating system (X.1), (X.2) reduces to integration of (X.4). Finding the solution of (X.4), we get an approximate solution of the initial problem from (X.3).

As will be shown below, determination of the sequence of functions A_1, B_1, u_1, v_1,... presents no difficulties in principle, but in view of the rapid complication of equations (X.6), it is usually possible only to construct the first few functions A_1, B_1, u_1, v_1,... Therefore, the problem of convergence of series (X.3) when the number of terms increases without bound is not considered in the method of asymptotic integration. However, the approximate solution consisting of l terms of series (X.3) has an asymptotic property: for sufficiently small ε, it approximates the exact solution of system (X.1), (X.2) with error $\sim \varepsilon^{l+1}$ on the time interval $t \sim \varepsilon^{-1}$

Let us go on to solution of the system (X.6). We expand the functions $X_1(\bar{x}, \bar{y})$, $Y_1(\bar{x}, \bar{y})$,... and the unknown functions $u_1(\bar{x}, \bar{y})$, $v_1(\bar{x}, \bar{y})$,... in double Fourier series. Writing them out in complex form, we get

$$\left.\begin{aligned} X_1 &= \sum_{k,s=-\infty}^{\infty} X_1^{(k,s)} \exp(ik\bar{y}_1 + is\bar{y}_2); \\ Y_1 &= \sum_{k,s=-\infty}^{\infty} Y_1^{(k,s)} \exp(ik\bar{y}_1 + is\bar{y}_2); \\ u_1 &= \sum_{k,s=-\infty}^{\infty} u_1^{(k,s)} \exp(ik\bar{y}_1 + is\bar{y}_2); \\ v_1 &= \sum_{k,s=-\infty}^{\infty} v_1^{(k,s)} \exp(ik\bar{y}_1 + is\bar{y}_2), \end{aligned}\right\} \qquad (X.7)$$

Here $X_1^{(k,s)}$, $Y_1^{(k,s)}$, $u_1^{(k,s)}$, $v_1^{(k,s)}$. are coefficients of the Fourier series for the functions X_1, Y_1, u_1, v_1, in particular,

$$X_1^{(k,s)} = \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} X_1(\bar{x},\bar{y}) e^{-i(k\bar{y}_1 + s\bar{y}_2)} d\bar{y}_1 \, d\bar{y}_2 .$$

The complex exponential form of the Fourier series (X.7) is equivalent to the ordinary expansion in sines and cosines. Let us substitute expansion (X.7) in the first and second equations of system (X.6). Equating the coefficients associated with identical harmonics, we get

$$A_1(\bar{x}) = \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} X_1(\bar{x},\bar{y}_1,\bar{y}_2) d\bar{y}_1 \, d\bar{y}_2 ; \qquad (X.8)$$

$$B_1(\bar{x}) = \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} Y_1(\bar{x},\bar{y}_1,\bar{y}_2)\,d\bar{y}_1 d\bar{y}_2; \quad (X.9)$$ /290

$$X_1^{(k,s)} = i(n_1 k + n_2 s)u_1^{(k,s)}; \quad (X.10)$$

$$Y_1^{(k,s)} = i(n_1 k + n_2 s)v_1^{(k,s)}. \quad (X.11)$$

Formulas (X.8) and (X.9) determine the functions $A_1(\bar{x})$, $B_1(\bar{x})$. If frequencies n_1 and n_2 are noncommensurate, i.e. if

$$n_1 k + n_2 s \neq 0 \quad (X.12)$$

for all integers m, n are unidentical to zero simultaneously, then the Fourier coefficients of the functions u_1, v_1 may be found from (X.10) and (X.11). The functions u_1, v_1 are determined with accuracy to an arbitrary constant.

In order to eliminate this ambiguity, additional requirements must be imposed on the functions u_1, v_1, e.g. the requirement that u_1 and v_1 must equal zero at the initial instant of motion, or requirement of absence of the first harmonic with respect to $\bar{y}$. The given selection of an additional condition has no effect on the accuracy of the approximate solution.

Let us require the functions $u_1(\bar{x}, \bar{y})$, $v_1(\bar{x}, \bar{y})$ to contain no zeroth harmonic with respect to $\bar{y}$, in other words, we shall define these functions so that the equalities

$$\int_0^{2\pi}\int_0^{2\pi} u_1(\bar{x},\bar{y})\,dy_1 dy_2 = 0; \quad \int_0^{2\pi}\int_0^{2\pi} v_1(\bar{x},\bar{y})\,d\bar{y}_1 d\bar{y}_2 = 0. \quad (X.13)$$

are satisfied.

Let us expand functions X_2, Y_2 in Fourier series and substitute the series in the second and third equations of system (X.7). We shall use solutions (X.8) - (X.11). Equating the coefficients associated with identical harmonics, we find

$$\left.\begin{aligned}A_2&=\frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}\left[\frac{\partial X_1(\bar{x},\bar{y})}{\partial\bar{x}}u_1(\bar{x},\bar{y})+\frac{\partial X_1(\bar{x},\bar{y})}{\partial\bar{y}}v_1(\bar{x},\bar{y})+X_2(\bar{x},\bar{y})\right]d\bar{y}_1\,d\bar{y}_2;\\ B_2&=\frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}\left[\frac{\partial Y_1(\bar{x},\bar{y})}{\partial\bar{x}}u_1(\bar{x},\bar{y})+\frac{\partial Y_1(\bar{x},\bar{y})}{\partial\bar{y}}v_1(\bar{x},\bar{y})+Y_2(\bar{x},\bar{y})\right]d\bar{y}_1\,d\bar{y}_2.\end{aligned}\right\}\qquad (X.13')$$

Continuing the process of successive determination of the functions we need, we may construct a solution of system (X.6) in any approximation under condition (X.12). We call the approximate solution x(t), y(t) a first-approximation solution if the resultant error has an order of $\sim\varepsilon$ on time interval $t\sim\varepsilon^{-1}$. To construct a first-approximation system, it is suf- /291
ficient to integrate the averaged system

$$\left.\begin{aligned}\dot{\bar{x}}&=\varepsilon A_1(\bar{x});\\ \dot{\bar{y}}&=n+\varepsilon B_1(\bar{x})\end{aligned}\right\}\qquad (X\ 14)$$

with initial data

$$\tau=0;\quad \bar{x}=x_0;\quad \bar{y}=y_0 \qquad (X.14')$$

and to set $x = \bar{x}$, $y = \bar{y}$.

The second-approximation solution approaches the exact solution of system (X.1), (X.2) with an error of the order of ε^2 on the interval $t\sim\varepsilon^{-1}$. To get the second approximation, it is necessary to find the functions u_1, v_1, A_1, B_1, A_2, B_2 and to solve the system

$$\left.\begin{aligned}\dot{\bar{x}}&=\varepsilon A_1(\bar{x})+\varepsilon^2A_2(\bar{x});\\ \dot{\bar{y}}&=n+\varepsilon B_1(\bar{x})+\varepsilon^2B_2(\bar{x}).\end{aligned}\right\}\qquad (X.14'')$$

If requirement (X.13) is imposed on functions u_1, v_1, then the initial conditions for (X.14) will take the form

$$\tau=0;\quad \bar{x}=x_0-\varepsilon u_1(x_0,y_0);\quad \bar{y}=y_0-\varepsilon v_1(x_0,y_0).$$

The solution of system (X.14") may be simplified if we seek the solution in the form

$$\left.\begin{aligned} \bar{x} &= \xi_1 + \varepsilon\eta_1; \\ \bar{y} &= \xi_2 + \varepsilon\eta_2, \end{aligned}\right\} \qquad (X.15)$$

where ξ_1, ξ_2 are the solutions of system (X.14) for $\bar{x}$ and $\bar{y}$, respectively.

Let us substitute (X.15) in (X.14") and expand the left-hand and right-hand members in a series with respect to ε. Dropping terms of order ε^2 and higher, we get a system for η_1, η_2:

$$\left.\begin{aligned} \dot{\eta}_1 &= \varepsilon\frac{\partial A_1}{\partial \bar{x}}\eta_1 + \varepsilon A_2(\xi_1); \\ \dot{\eta}_2 &= \varepsilon\frac{\partial B_1}{\partial \bar{x}}\eta_1 + \varepsilon B_2(\xi_1). \end{aligned}\right\} \qquad (X.16)$$

Thus, finding the unknown functions η_1 and η_2 reduces to solving a linear nonhomogeneous system of differential equations

If $\partial A_1/\partial\bar{x} = \partial B_1/\partial\bar{x} = 0$, then η_1 and η_2 are found with the aid of quadratures:

$$\begin{aligned} \eta_1 &= \varepsilon\int A_2(\xi_1)\,dt; \\ \eta_2 &= \varepsilon\int B_2(\xi_1)\,dt. \end{aligned}$$ /292

Returning to the original variables, we get a final solution for the problem in the second approximation:

$$\begin{aligned} \bar{x} &= \xi_1 + \varepsilon\eta_1 + \varepsilon u_1(\xi_1, \xi_2); \\ \bar{y} &= \xi_2 + \varepsilon\eta_2 + \varepsilon v_1(\xi_1, \xi_2). \end{aligned}$$

We shall say that resonance takes place if frequencies n_1 and n_2 are commensurate. In this case, there also exist integral mutually simple numbers k and s such that the difference

$$kn_1 - sn_2 \approx O(\varepsilon).$$

In the resonance case, condition (X.13) is violated. Functions u_1, v_1 may not be determined from system (X.6).

For calculating resonance conditions, let us introduce the new variable z--phase shift:

$$z = y_1 k - y_2 s. \tag{X.17}$$

Eliminating the rapidly changing variable y_1 from system (X.1), (X.2) by formula (X.17), we get the system

$$\left.\begin{aligned} \dot{x} &= \varepsilon X_1(x, z, y_2) + \varepsilon^2 X_2(x, z, y_2) + \dots ; \\ \dot{z} &= n_1 k - n_2 s + \varepsilon k Y_{11}(x, z, y_2) - \varepsilon s Y_{12}(x, y_2, z) + \dots ; \\ \dot{y}_2 &= n_2 + \varepsilon Y_{12}(x, z, y_2) + \dots . \end{aligned}\right\} \tag{X.18}$$

System (X.18) again has the standard form. In this system, x and z are slowly changing variables while y_2 is a rapidly changing variable. If the functions X_1, Y_1,... depend on time (nonautonomous system), then by introducing the new variable W which satisfies the equation $\dot{W} = 1$, and substituting W for time τ in the functions X_1, Y_1,..., we again reduce the system to standard form (X.1), (X.2).

Applying now the system of computation outlined above, we solve the problem of asymptotic integration of system (X.1), (X.2).

A more detailed exposition of the averaging method and a strict validation may be found in [64 - 67].

REFERENCES

1. Subbotin, M. F., *"Kurs nebesnoy mekhaniki"*, [A Course in Celestial Mechanics], Vol. II, *State Technical Press*, Moscow, 1947.
2. Duboshin, G. N., *"Nebesnaya mekhanika. Osnovnyye zadachi i metody"*, [Celestial Mechanics, Fundamental Problems and Methods], *State Physics and Mathematics Press*, Moscow, 1963.
3. El'yasberg, P. Ye., *"Vvedeniye v teoriyu poleta iskusstvennykh sputnikov Zemli"*, [Introduction to the Theory of Flight of Artificial Earth Satellites], *"Nauka" Press*, Moscow, 1965.
4. Duboshin G. N., and D. Ye. Okhotsimskiy, "Some Problems of Astrodynamics and Celestial Mechanics", Kosmicheskiye Issledovaniya, Vol. 1, No. 2. 1963.
5. Blitzer, S., "Lunar-Solar Perturbations of an Earth Satellite", *Americ. Journal of Phys.*, Vol. 27, No. 9, 1959.
6. Lidov, M. L., "Evolution of the Orbits of Artificial Satellites of Planets Under the Effect of Gravitational Perturbations of External Bodies" in the collection *"Iskusstvennyye sputniki Zemli*, No. 8, *AN SSSR Press*, Moscow, 1961.
7. Kozai, J. and Ch. Whitney, "Anticipated Orbital Perturbations of Satellite 1959 Delta Two", Spec. Rept. *Astrophys. Observ. Smithsonian Inst.*, No. 30, 1959.
8. Chernyy, G. G., *"Techeniye gazov s bol'shoy sverkhzvukovoy skorost'yu"*, [Flow of Gases with High Supersonic Velocity], *Physics and Mathematics Press*, Moscow, 1959.
9. Rowell, L. M. and M. C. Smith, "Secular Variations in the Inclination of the Orbit of Earth Satellite (1957 β) and Air Drag", Xth Internat. Astronaut. Congr., London -- 1959, Vol. 1, Vienna, 1960.
10. Sterne, T., "Effect of the Rotation of a Planetary Atmosphere Upon the Orbit of a Close Satellite," *ARS Journal*, Vol. 29, No. 10, 1959.
11. Mikhnevich, V. V., "Variations in Atmospheric Density at Altitudes Greater than 200 km," in the collection *Iskustvennyye sputniki Zemli*, No.17, *AN SSSR Press*, Moscow, 1963.
12. Kolegov, G. A., "Variations in the Density of the Upper Atmosphere", in the collection *Iskusstvennyye sputniki Zemli*, No. 4, *AN SSSR Press*, 1960.
13. Okhotsimskiy, D. Ye., T. M. Eneyev and G. P. Taratynova, "Determining the Lifetime of an Artificial Earth Satellite and Studying the Secular Perturbations of its Orbit," *Uspekhi fiz. Nauk*, Vol. LXIII, No. 1a, 1957.
14. Taratynova, G. P., "On the Motion of an Artificial Earth Satellite in the Central Gravitational Field of the Earth in the Presence of Atmospheric Resistance," *Uspekhi fiz. Nauk*, Vol. LXIII, No. 1a,1957.
15. Taratynova, G. P., "Methods for Numerical Solution of Equations in Finite Differences," in the collection *Iskusstvennyye sputniki Zemli*, No. 4, *AN SSSR Press*, Moscow, 1960.
16. El'yasberg, P. Ye., "Secular Variations of Orbital Elements as a Function of Air Resistance," in the collection *Iskusstvennyye sputniki Zemli*, No. 3, *AN SSSR Press*, Moscow, 1959.

17. El'yasberg, P. Ye., " Approximate Formulas for Dtermining the Lifetime of Artificial Earth Satellites," *Kosmicheskiye issledovaniya*, Vol. II, No. 2, 1964.
18. Mitra, S. K., *"Verkhnyaya atmosfera,"* [The Upper Atmosphere], *Foreign Literature Press*, Moscow, 1955.
19. Zhongolovich, I. D., "Some Formulas Relating to Motion of a Material Point in the Gravitational Field of an Equipotential Ellipsoid of Revolution," *Byulleten' ITA AN SSSR*, Vol. 7, No. 7 (90), 1960.
20. Beletskiy, V. V., "Evolution of the Rotation of a Dynamically Symmetric Satellite," *Kosmicheskiye issledovaniya*, Vol. 1, No. 3, 1963.
21. Ginzburg, V. L., "Experimental Verification of the General Theory of Relativity," *Uspekhi fiz. Nauk*, Vol. 59, No. 1, 1956.
22. Tharraty, Y., "Relativity Advances of the Perigee of Artificial Earth Satellites," *Tenth Internat. Astronaut. Congr. London, 1959*, Vol. 2, Wien, 1960.
23. Bogorodskiy, A. F., "On Relativistic Effects in the Motion of Artificial Earth Satellites," *Publikatsii Kiyevskoy astronomicheskoy observatorii*, No. 11, 1962.
24. Musen, P., "The Influence of the Solar Radiation Pressure on the Motion of an Artificial Satellite," *J. Geophys. Res.*, Vol. 65, No. 5, 1960.
25. Roberson, R. E., "Attitude Control of a Satellite Vehicle, an Outline of The Problems," *VIII-th Internat. Astronaut. Congr., Barcelona, 1957*, Wien, 1958.
26. Roberson, R. E., "Attitude Control of Satellite and Space Vehicles," *Advances Space Sci.*, Vol. 2, *Acad. Press*, N. Y. - London, 1960.
27. Koskela, P. E., "Orbital Effects of Solar Radiation Pressure on an Earth Satellite," *Journal of Astronaut. Sci.*, Vol. 9, No. 3, 1962.
28. Polyakhova, Ye. N., "The Pressure of Light and the Motion of an Earth Satellite," *Byulleten' ITA AN SSSR*, Vol. 9, No. 1, 1963.
29. Parkinson, R. W., N. M. Jones and J. S. Shapiro, "The Effect of Solar Radiation Pressure on Earth Satellite Orbits," *Science*, Vol. 131, No. 3404, 1960.
30. Muhleman, Hudson, Holdridge, Carpenter, "Observed Solar Pressure Perturbations of 'Echo-I'," *Science*, Vol. 132, No. 3438, 1960.
31. Doboshin, G. N., *"Teoriya prityazheniya,"* [Theory of Gravitation], *Physics and Mathematics Press*, Moscow, 1961.
32. Zhongolovich, I. D., "Potential of Terrestrial Attraction," *Byulleten' ITA AN SSSR*, Vol. VI, No. 8(81), 1957.
33. Jeffreys, H., "The Earth, Its Origin, History and Phys. Constit.," 3d Ed., *Cambr. University Press* (there is a Russian translation), 1952.
34. Poincare, H., "Gravity Measurements and Geodesy," *Bulletin Astronomique*, XVIII, 1901 (Russian translation in the collection edited by A. A. Mikhaylov,"*Stat' i o sile tyazhesti i figure Zemli*", *Geodetic Literature Press*, Moscow, 1961).
35. Zhongolovich, I. D., "The Outer Gravitational Field of the Earth and Fundamental Constants Related to It," *Trudy ITA*, No. III, 1952.
36. Brovar, V. V., V. A. Magnitskiy and B. P. Shimbirev, *"Teoriya figury Zemli,"* [Theory of the Figure of the Earth], *Geodetic Press*, Moscow, 1961.
37. Grushinskiy, N. P.,*"Teoriya figury Zemli,"* [Theory of the Figure of the

Earth], *Physics and Mathematics Press*, Moscow, 1963.
38. Zagrebin, D. V., "Theory of the Regularized Geoid," *Trudy ITA*, No. 1, pp. 87-220, 1952.
39. Molodenskiy, M. S., V. F. Yeremeyev and M. I. Yurkina, "Methods of Studying the External Gravitational Field and Figure of the Earth," *Trudy TsNIIGAiK*, No. 131, Moscow, 1960.
40. Pellinen, L. P., "Effect of Topographic Masses on Derivation of the Characteristics of the Earth's Gravitational Field," *Trudy TsNIIGAiK*, No. 145, 1962. /295
41. Kozai, J., "The Gravitational Field of the Earth Derived from Motions of the Satellites," *Astron. Journal*, No. 1, 1960.
42. Kaula, W. M., "A Geoid and World Geodetic System Based on a Combination of Gravimetric, Astrogeodetic and Satellite Data," *Journal of Geophysical Research*, Vol. 66, No. 6, June 1961
43. Kaula, W. M., "Statistical and Harmonic Analysis of Gravity," *Journal of Geophysical Research*, Vol. 64, No. 12, 1959.
44. Sretenskiy, L. N., *"Teoriya n'yutonovskogo potentsiala,"* [Theory of the Newtonian Potential], *State Technical Press*, Moscow, 1946.
45. Subbotin, M. F., *"Kurs nebesnoy mekhaniki,"* [A Course in Celestial Mechanics], Vol. III, *State Technical Press*, Moscow, 1949.
46. Samoylovich, G. V., "System of Parameters for Describing the Orbits of Space Vehicles," in the collection *Iskusstvennyye sputniki Zemli*, No. 16, *AN SSSR Press*, Moscow, 1963.
47. Kalil, F. and F. Martikan, "Deviation of Nodal Period of an Earth Satellite and Comparisons of Several First-Order Secular Oblateness Results," *AIAA Journal*, Vol. 1, No. 9, 1963.
48. Samoylovich, G. V., "Motion of an Artificial Satellite of the Nonspherical Earth," in the collection *Iskusstvennyye sputniki Zemli*, Vol. 16, *AN SSSR Press*, Moscow, 1963.
49. Demin, V. G., "Application of V. V. Rumyantsev's Theorem on Stability in Relationship to Some of the Variables in Problems of Celestial Mechanics," *Kosmicheskiye issledovaniya*, Vol. II, No. 5, 1964.
50. Demin, V. G., "On the Stability of Satellite Orbits in the Case of Constantly Acting Perturbations," *Kosmicheskiye issledovaniya*, Vol. II, No. 5, 1964.
51. Samoylovich, G. V., "Effect of Orbital Parameters on the Disturbed Motion of an Artificial Earth Satellite," in the collection *Iskusstvennyye Sputniki Zemli*, No. 16, *AN SSSR Press*, Moscow, 1963.
52. Proskurin, V. F. and Yu. V. Batrakov, "Disturbances in the Motion of an Artificial Satellite due to Flattening of the Earth," *Byulleten' ITA AN SSSR*, Vol. 7, No. 7(90), 1960.
53. Proskurin, V. F. and Yu. V. Batrakov, "First-Order Disturbances in the Motion of an Artificial Satellite due to Flattening of the Earth," in the collection *Iskusstvennyye sputniki Zemli*, No. 3, *AN SSSR Press*, Moscow, 1959.
54. Kozai, J., "The Motion of a Close Earth Satellite," *The Astronomical Journal*, Vol. 64, No. 9, 1959.
55. Kozai, J., "Second-Order Solution of Artificial Satellite Theory Without Air Drag," *The Astronomical Journal*, Vol. 67, No. 7, 1962.

56. Garfinkel, B., "On the Motion of a Satellite of an Oblate Planet," *The Astronomical Journal*, Vol. 63, No. 3, 1958.
57. Garfinkel, B., "Formulae for Predicting the Position of an Artificial Satellite," *The Astronomical Journal*, Vol. 64, No. 7, 1959.
58. Garfinkel, B., "The Orbit of a Satellite of an Oblate Planet," *The Astronomical Journal*, Vol. 64, No. 9, 1959.
59. Antony, M. L. and G. E. Forsdick, "Satellite Motions About an Oblate Planet," *Journal Aeronautical Sciences*, Vol. 28, No. 10, 1961.
60. Lakhtin, L. M., "*Svobodnoye dvizheniye v pole zemnogo sferoida*", [Free Motion in the Field of a Terrestrial Spheroid], *Physics and Mathematical Press*, Moscow, 1963.
61. Malkin, I. G., "*Metody Lyapunova i Puankare v teorii nelineynykh kolebaniy,*" [The Lyapunov and Poincare Methods in the Theory of Nonlinear Oscillations,] *State Technical Press*, Moscow, 1949. /296
62. Poincare, H., "*Les methodes nouvelles de la mecanique celeste,*" [New Methods of Celestial Mechanics], Paris, 1879.
63. Golubev, V. V., "Lectures on the Analytical Theory of Differential Equations," *State Technical Press*, Moscow, 1950.
64. Bogolyubov, N. N. and Yu. A. Mitropol'skiy, "*Asimptoticheskiye metody v teorii nelineynykh kolebaniy,*" [Asymptotic Methods in the Theory of Nonlinear Oscillations], *Physics and Mathematics Press*, Moscow, 1938.
65. Bogolyubov, N. N. and D. N. Zubarev, "The Method of Asymptotic Approximation for Systems with a Rapidly Rotating Phase and its Application to the Motion of Charged Particles in a Magnetic Field,", *Ukr. matematich. zhurnal*, Vol. 7, No. 1, 1955.
66. Volosov, V. M., "Some Types of Computations in the Theory of Nonlinear Oscillations Associated with Averaging," *Zhurnal vychisl. matem. i matem. fiz.*, Vol. 3, No. 1, 1963.
67. Volosov, V. M., "Averaging in Systems of Ordinary Differential Equations," *Uspekhi matem. Nauk*, Vol. XVII, No. 6, 1962.
68. Moiseyev, N. N., "Methods of Nonlinear Mechanics in the Problems of Dynamics of Satellites," *Reports of the XIII International Congress of Aeronautics*, Varna, 1962.
69. Akhiyezer, N. I., "*Elementy teorii ellipticheskikh funktsiy,*" [Elements of the Theory of Elliptic Functions], *State Technical Press*, Moscow, 1948.
70. Kislik, M. D., "Motion of an Artificial Satellite in the Normal Gravitational Field of the Earth," in the collection *Iskusstvennyye sputniki Zemli*, No. 4, *AN SSSR Press*, Moscow, 1960.
71. Kislik, M. D., "Analysis of the Integrals of the Equations of Motion of an Artificial Satellite in the Normal Gravitational Field of the Earth", In the collection *Iskusstvennyye sputniki Zemli*, No. 13, *AN SSSR Press*, Moscow, 1962.
72. Vinti, J. P., "New Model of Solution for Unretarded Satellite Orbits," *Journal Res. of Nat. Bur. Standards*, Vol. 62B, No. 2, 1959.
73. Yakoby, K., "*Lektsii po dinamike,*" [Lectures on Dynamics],*GTTI Press*, Moscow, 1938.

74. Orlov, A. A., "Nearly Circular Periodic Motions of a Material Point Under the Effect of the Newtonian Attraction of a Spheroid," *Soobshcheniya GAISh*, No. 88-89, 1953.
75. Orlov, A. A., "On Nearly Periodic Motions of a Material Point in the Gravitational Field of a Spheroid," *Trudy GAISh*, Vol. XXIV, 1954.
76. Orlov, A. A., "Calculation of Terms of Second Order Relative to Oblateness of the Earth in the Coordinates of an Artificial Earth Satellite," *Byulleten' ITA AN SSSR*, Vol. 7, No. 7, 1960.
77. Orlov, A. A., "Effect Which the Spheroidal Shape of a Planet has on Motion of its Satellite," *Soobshcheniya GAISh*, No. 112, 1961.
78.. Aksenov, Ye. P., Ye. A. Grebennikov and V. G. Demin, "General Solution of the Problem of Motion of an Artificial Satellite in the Normal Gravitational Field of the Earth," in the collection *Iskusstvennyye sputniki Zemli*, No. 8, *AN SSSR Press*, Moscow, 1961.
79. Aksenov, Ye. P., Ye. A. Grebennikov and V. G. Demin, "Qualitative Analysis of Forms of Motion in the Problem of Motion of an Artificial Satellite in the Normal Gravitational Field of the Earth," In the collection *Iskusstvennyye sputniki Zemli*, No. 16, *AN SSSR Press*, Moscow, 1963.
80. Aksenov, Ye. P., "Plotting an Intermediate Orbit for an Artificail Earth /297 Satellite," *Vestnik MGU, seriya fizika, astronomiya*, No. 3, 1965.
81. Yatsunskiy, I. M., "The Effect Which Geophysical Factors Have on Satellite Motion," *Uspekhi fiz. Nauk*, Vol. 63, No. 1a, 1957.
82. Van der Varden, B. L, *"Matematicheskaya statistika,"* [Mathematical Statistics], *Foreign Literature Press*, Moscow, 1960. [Russian Tr.]
83. Arley, N. and K. R. Bukh, *"Vedeniye v teoriyu veroyatnosti i matematicgesjyty statistiku,"* [Introduction to Probability Theory and Mathematical Statistics], *State Technical Press*, Moscow, 1951.
84. Keldysh, M. V., "On Solubility and Stability of the Dirichlet Problem," *Uspekhi matem. Nauk*, No. 8, 1940.
85. Tikhonov, A. N. and A. A. Samarskiy, *"Uravneniya matematicheskoy fiziki,"* [Equations of Mathematical Physics], *GTTI Press*, Moscow, 1951.
86. Kurant, R. and D. Gil'bert, *"Metody matematicheskoy fiziki, t. l,"* [Methods of Mathematical Physics, Vol. 1], *GTTI Press*, Moscow, 1951.
87. Uitteker, E. T. and Dzh. N. Vatson, *(Sk), "Kurs sovremennogo analiza, t. II,"* [Course in Modern Analysis, Vol. II], *State Physics and Mathematics Press*, Moscow, 1963.
88. Tables of Associated Legendre Functions, Library of Mathematical Tables, No. 14, *VTs AN SSSR*, Moscow, 1962.
89. Sturms, F. M., "Nodal Period for a Circular Earth Satellite," *ARS Journal*, Vol. 32, No. 7, 1962.
90. Blitzer, L., E.M. Boughton, G. Kang and R. M. Page, "Effect of Ellipticity of the Equator on 24-hour Nearly Circular Satellite Orbits," *Journal of Geophys. Res.*, Vol. 67, 1962.
91. Blitzer, L., G. Kang and J. B. McGuire, "The Perturbed Motior of 24-hour Satellite due to Equatorial Ellipticity," *Journ. of Geophys. Res.*, Vol. 68, 1963.
92. Perkins, F. M., "Flight Mechanics of the 24-Hour Satellites," *AIAA Journal*, Vol. 1, No. 4, 1963.

93. Samoylovich, G. V., "Evolution of the Orbit of a Circular Satellite," *Kosmicheskiye issledovaniya*, Vol. II, No. 2, 1964.
94. Aksenov, Ye. P., "Determining the Orbital Elements of a Satellite from Initial Data," *Trudy GAISh*, No. 35, pp. 36-42, 1966.
95. King Hele, D. G. and G. E. Cook, "The Even Zonal Harmonics of the Earth's Gravitational Potential," *Royal Aircraft Establishment, Tech. Note*, No. 68, July 1964.
96. Kozai, J., "New Determination of Zonal Harmonics Coefficients in the Earth Gravitational Potential," Rept., *Cospar*, 1964 (Firence).

Translated for the National Aeronautics and Space Administration under contract No. NASw-1695 by Techtran Corporation, P. O. Box 729, Glen Burnie, Md. 21061.

NASA-Langley, 1968 —— 30 F-539

"The aeronautical and space activities of the United States shall be conducted so as to contribute . . . to the expansion of human knowledge of phenomena in the atmosphere and space. The Administration shall provide for the widest practicable and appropriate dissemination of information concerning its activities and the results thereof."

—NATIONAL AERONAUTICS AND SPACE ACT OF 1958

NASA SCIENTIFIC AND TECHNICAL PUBLICATIONS

TECHNICAL REPORTS: Scientific and technical information considered important, complete, and a lasting contribution to existing knowledge.

TECHNICAL NOTES: Information less broad in scope but nevertheless of importance as a contribution to existing knowledge.

TECHNICAL MEMORANDUMS: Information receiving limited distribution because of preliminary data, security classification, or other reasons.

CONTRACTOR REPORTS: Scientific and technical information generated under a NASA contract or grant and considered an important contribution to existing knowledge.

TECHNICAL TRANSLATIONS: Information published in a foreign language considered to merit NASA distribution in English.

SPECIAL PUBLICATIONS: Information derived from or of value to NASA activities. Publications include conference proceedings, monographs, data compilations, handbooks, sourcebooks, and special bibliographies.

TECHNOLOGY UTILIZATION PUBLICATIONS: Information on technology used by NASA that may be of particular interest in commercial and other non-aerospace applications. Publications include Tech Briefs, Technology Utilization Reports and Notes, and Technology Surveys.

Details on the availability of these publications may be obtained from:

SCIENTIFIC AND TECHNICAL INFORMATION DIVISION

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

Washington, D.C. 20546